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Boolean Domains

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Summary. BOOLE DOMAIN is a SET DOMAIN that is closed under union and difference. This condition is equivalent to being closed under symmetric difference and one of the following operations: union, intersection or difference. We introduce the set of all finite subsets of a set A, denoted by Fin A. The mode Finite Subset of a set A is introduced with the mother type: Element of Fin A. In consequence, "Finite Subset of ..." is an elementary type, therefore one may use such types as "set of Finite Subset of A", "[(Finite Subset of A), Finite Subset of A]", and so on. The article begins with some auxiliary theorems that belong really to [5] or [1] but are missing there. Moreover, bool A is redefined as a SET DOMAIN, for an arbitrary set A.

The articles [4], [5], [3], and [2] provide the notation and terminology for this paper. In the sequel X, Y will denote objects of the type set. The following propositions are true:

$$X \text{ misses } Y \textbf{ implies } X \setminus Y = X \ \&\ Y \setminus X = Y, \tag{1}$$

$$X \text{ misses } Y \textbf{ implies } (X \cup Y) \setminus Y = X \ \&\ (X \cup Y) \setminus X = Y, \tag{2}$$

$$X \cup Y = X \dot{-} (Y \setminus X), \tag{3}$$

$$X \cup Y = X \dot{-} Y \dot{-} X \cap Y, \tag{4}$$

$$X \setminus Y = X \dot{-} (X \cap Y), \tag{5}$$

$$X \cap Y = X \dot{-} Y \dot{-} (X \cup Y), \tag{6}$$

$$(\textbf{for } x \textbf{ being } \text{set } \textbf{st } x \in X \textbf{ holds } x \in Y) \textbf{ implies } X \subseteq Y. \tag{7}$$

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Let us consider X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\text{bool}\, X \qquad \text{is} \qquad \text{SET\_DOMAIN}\,.$$

The following proposition is true

(8) **for** Y **being** Element **of** bool X **holds** $Y \subseteq X$.

The mode

BOOLE_DOMAIN ,

which widens to the type SET_DOMAIN, is defined by

for X,Y **being** Element **of it holds** $X \cup Y \in$ **it** & $X \setminus Y \in$ **it** .

The following proposition is true

(9) **for** A **being** SET_DOMAIN **holds** A **is** BOOLE_DOMAIN
iff for X,Y **being** Element **of** A **holds** $X \cup Y \in A$ & $X \setminus Y \in A$.

In the sequel A will denote an object of the type BOOLE_DOMAIN. One can prove the following propositions:

(10) $X \in A$ & $Y \in A$ **implies** $X \cup Y \in A$ & $X \setminus Y \in A$,

(11) X **is** Element **of** A & Y **is** Element **of** A **implies** $X \cup Y$ **is** Element **of** A,

(12) X **is** Element **of** A & Y **is** Element **of** A **implies** $X \setminus Y$ **is** Element **of** A.

The arguments of the notions defined below are the following: A which is an object of the type reserved above; X, Y which are objects of the type Element **of** A. Let us note that it makes sense to consider the following functors on restricted areas. Then

$X \cup Y$ is Element **of** A,

$X \setminus Y$ is Element **of** A.

The following propositions are true:

(13) X **is** Element **of** A & Y **is** Element **of** A **implies** $X \cap Y$ **is** Element **of** A,

(14) X **is** Element **of** A & Y **is** Element **of** A **implies** $X \dot{-} Y$ **is** Element **of** A,

(15) **for** A **being** SET_DOMAIN **st**
for X,Y **being** Element **of** A **holds** $X \dot{-} Y \in A$ & $X \setminus Y \in A$
holds A **is** BOOLE_DOMAIN ,

(16) $$\textbf{for } A \textbf{ being } \mathrm{SET\_DOMAIN} \textbf{ st}$$
$$\textbf{for } X,Y \textbf{ being } \mathrm{Element} \textbf{ of } A \textbf{ holds } X \dot{-} Y \in A \ \&\ X \cap Y \in A$$
$$\textbf{holds } A \textbf{ is } \mathrm{BOOLE\_DOMAIN}\,,$$

(17) $$\textbf{for } A \textbf{ being } \mathrm{SET\_DOMAIN} \textbf{ st}$$
$$\textbf{for } X,Y \textbf{ being } \mathrm{Element} \textbf{ of } A \textbf{ holds } X \dot{-} Y \in A \ \&\ X \cup Y \in A$$
$$\textbf{holds } A \textbf{ is } \mathrm{BOOLE\_DOMAIN}\,.$$

The arguments of the notions defined below are the following: A which is an object of the type reserved above; X, Y which are objects of the type Element **of** A. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$X \cap Y \quad \text{is} \quad \mathrm{Element} \textbf{ of } A,$$

$$X \dot{-} Y \quad \text{is} \quad \mathrm{Element} \textbf{ of } A.$$

We now state four propositions:

(18) $$\emptyset \in A,$$

(19) $$\emptyset \textbf{ is } \mathrm{Element} \textbf{ of } A,$$

(20) $$\mathrm{bool}\, A \textbf{ is } \mathrm{BOOLE\_DOMAIN}\,,$$

(21) $$\textbf{for } A,B \textbf{ being } \mathrm{BOOLE\_DOMAIN} \textbf{ holds } A \cap B \textbf{ is } \mathrm{BOOLE\_DOMAIN}\,.$$

In the sequel A, B will denote objects of the type set. Let us consider A. The functor

$$\mathrm{Fin}\, A,$$

with values of the type BOOLE_DOMAIN, is defined by

$$\textbf{for } X \textbf{ being } \mathrm{set} \textbf{ holds } X \in \textbf{it iff } X \subseteq A \ \&\ X \ \mathrm{is\_finite}\,.$$

The following propositions are true:

(22) $$B \in \mathrm{Fin}\, A \textbf{ iff } B \subseteq A \ \&\ B \ \mathrm{is\_finite}\,,$$

(23) $$A \subseteq B \textbf{ implies } \mathrm{Fin}\, A \subseteq \mathrm{Fin}\, B,$$

(24) $$\mathrm{Fin}\,(A \cap B) = \mathrm{Fin}\, A \cap \mathrm{Fin}\, B,$$

(25) $$\mathrm{Fin}\, A \cup \mathrm{Fin}\, B \subseteq \mathrm{Fin}\,(A \cup B),$$

(26) $$\mathrm{Fin}\, A \subseteq \mathrm{bool}\, A,$$

(27) $$A \ \mathrm{is\_finite} \textbf{ implies } \mathrm{Fin}\, A = \mathrm{bool}\, A,$$

(28) $$\mathrm{Fin}\, \emptyset = \{\emptyset\}.$$