FORMALIZED MATHEMATICS
Number 1, January 1990
Université Catholique de Louvain

Functions from a Set to a Set

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Summary. The article is a continuation of [1]. We define the following concepts: a function from a set X into a set Y, denoted by "Function of X,Y", the set of all functions from a set X into a set Y, denoted by Funcs(X,Y), and the permutation of a set (mode Permutation of X, where X is a set). Theorems and schemes included in the article are reformulations of the theorems of [1] in the new terminology. Also some basic facts about functions of two variables are proved.

The notation and terminology used in this paper are introduced in the following articles: [2], [3], and [1]. For simplicity we adopt the following convention: P, Q, X, Y, $Y1$, $Y2$, Z will denote objects of the type set; x, $x1$, $x2$, y, $y1$, $y2$, z, $z1$, $z2$ will denote objects of the type Any. Let us consider X, Y. Assume that the following holds

$$Y = \emptyset \textbf{ implies } X = \emptyset.$$

The mode

$$\text{Function } \textbf{of } X, Y,$$

which widens to the type Function, is defined by

$$X = \text{dom}\, \textbf{it} \ \& \ \text{rng}\, \textbf{it} \subseteq Y.$$

Next we state several propositions:

(1) $$(Y = \emptyset \textbf{ implies } X = \emptyset) \textbf{ implies for } f \textbf{ being } \text{Function} \\ \textbf{holds } f \textbf{ is } \text{Function } \textbf{of } X, Y \textbf{ iff } X = \text{dom}\, f \ \& \ \text{rng}\, f \subseteq Y,$$

(2) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \\ \textbf{st } Y = \emptyset \textbf{ implies } X = \emptyset \textbf{ holds } X = \text{dom}\, f \ \& \ \text{rng}\, f \subseteq Y,$$

(3) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } f \textbf{ is } \text{Function } \textbf{of } \text{dom}\, f, \text{rng}\, f,$$

[1]Supported by RPBP.III-24.C1.

ISSN 0777-4028

(4) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{st } \operatorname{rng} f \subseteq Y \textbf{ holds } f \textbf{ is } \text{Function } \textbf{of } \operatorname{dom} f, Y,$$

(5) $$\textbf{for } f \textbf{ being } \text{Function} \\ \textbf{st } \operatorname{dom} f = X \ \& \ \textbf{for } x \textbf{ st } x \in X \textbf{ holds } f.x \in Y \textbf{ holds } f \textbf{ is } \text{Function } \textbf{of } X, Y,$$

(6) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ st } Y \neq \emptyset \ \& \ x \in X \textbf{ holds } f.x \in \operatorname{rng} f,$$

(7) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ st } Y \neq \emptyset \ \& \ x \in X \textbf{ holds } f.x \in Y,$$

(8) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \\ \textbf{st } (Y = \emptyset \textbf{ implies } X = \emptyset) \ \& \ \operatorname{rng} f \subseteq Z \textbf{ holds } f \textbf{ is } \text{Function } \textbf{of } X, Z,$$

(9) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \\ \textbf{st } (Y = \emptyset \textbf{ implies } X = \emptyset) \ \& \ Y \subseteq Z \textbf{ holds } f \textbf{ is } \text{Function } \textbf{of } X, Z.$$

In the article we present several logical schemes. The scheme *FuncEx1* deals with a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } f \textbf{ being } \text{Function } \textbf{of } \mathcal{A}, \mathcal{B} \textbf{ st for } x \textbf{ st } x \in \mathcal{A} \textbf{ holds } \mathcal{P}[x, f.x]$$

provided the parameters satisfy the following conditions:

- $\textbf{for } x \textbf{ st } x \in \mathcal{A} \textbf{ ex } y \textbf{ st } y \in \mathcal{B} \ \& \ \mathcal{P}[x, y],$
- $\textbf{for } x, y1, y2 \textbf{ st } x \in \mathcal{A} \ \& \ \mathcal{P}[x, y1] \ \& \ \mathcal{P}[x, y2] \textbf{ holds } y1 = y2.$

The scheme *Lambda1* concerns a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a unary functor $\mathcal{F}$ and states that the following holds

$$\textbf{ex } f \textbf{ being } \text{Function } \textbf{of } \mathcal{A}, \mathcal{B} \textbf{ st for } x \textbf{ st } x \in \mathcal{A} \textbf{ holds } f.x = \mathcal{F}(x)$$

provided the parameters satisfy the following condition:

- $\textbf{for } x \textbf{ st } x \in \mathcal{A} \textbf{ holds } \mathcal{F}(x) \in \mathcal{B}.$

Let us consider X, Y. The functor

$$\text{Funcs}\,(X, Y),$$

yields the type set and is defined by

$$x \in \textbf{it iff ex } f \textbf{ being } \text{Function } \textbf{st } x = f \ \& \ \operatorname{dom} f = X \ \& \ \operatorname{rng} f \subseteq Y.$$

We now state a number of propositions:

(10) $$\textbf{for } F \textbf{ being } \text{set } \textbf{holds } F = \text{Funcs}\,(X, Y) \textbf{ iff for } x \\ \textbf{holds } x \in F \textbf{ iff ex } f \textbf{ being } \text{Function } \textbf{st } x = f \ \& \ \operatorname{dom} f = X \ \& \ \operatorname{rng} f \subseteq Y,$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ st } Y=\emptyset \textbf{ implies } X=\emptyset \textbf{ holds } f \in \mathrm{Funcs}(X,Y), \tag{11}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,X \textbf{ holds } f \in \mathrm{Funcs}(X,X), \tag{12}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } \emptyset,X \textbf{ holds } f \in \mathrm{Funcs}(\emptyset,X), \tag{13}$$

$$X \neq \emptyset \textbf{ implies } \mathrm{Funcs}(X,\emptyset)=\emptyset, \tag{14}$$

$$\mathrm{Funcs}(X,Y)=\emptyset \textbf{ implies } X\neq\emptyset \;\&\; Y=\emptyset, \tag{15}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ st } Y\neq\emptyset \;\&\; \textbf{for } y \textbf{ st } y\in Y \textbf{ ex } x \textbf{ st } x\in X \;\&\; y=f.x \textbf{ holds } \mathrm{rng}\, f=Y, \tag{16}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ st } y\in Y \;\&\; \mathrm{rng}\, f=Y \textbf{ ex } x \textbf{ st } x\in X \;\&\; f.x=y, \tag{17}$$

$$\textbf{for } f1,f2 \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ st } Y\neq\emptyset \;\&\; \textbf{for } x \textbf{ st } x\in X \textbf{ holds } f1.x=f2.x \textbf{ holds } f1=f2, \tag{18}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y,Z \textbf{ st } (Z=\emptyset \textbf{ implies } Y=\emptyset) \;\&\; (Y=\emptyset \textbf{ implies } X=\emptyset) \textbf{ holds } g\cdot f \textbf{ is } \mathrm{Function} \textbf{ of } X,Z, \tag{19}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y,Z \textbf{ st } Y\neq\emptyset \;\&\; Z\neq\emptyset \;\&\; \mathrm{rng}\, f=Y \;\&\; \mathrm{rng}\, g=Z \textbf{ holds } \mathrm{rng}(g\cdot f)=Z, \tag{20}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y,Z \textbf{ st } Y\neq\emptyset \;\&\; Z\neq\emptyset \;\&\; x\in X \textbf{ holds } (g\cdot f).x=g.(f.x), \tag{21}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ st } Y\neq\emptyset \textbf{ holds } \mathrm{rng}\, f=Y \textbf{ iff for } Z \textbf{ st } Z\neq\emptyset \textbf{ for } g,h \textbf{ being } \mathrm{Function} \textbf{ of } Y,Z \textbf{ st } g\cdot f=h\cdot f \textbf{ holds } g=h, \tag{22}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ st } Y=\emptyset \textbf{ implies } X=\emptyset \textbf{ holds } f\cdot(\mathrm{id}\, X)=f \;\&\; (\mathrm{id}\, Y)\cdot f=f, \tag{23}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y,X \textbf{ st } Y\neq\emptyset \;\&\; f\cdot g=\mathrm{id}\, Y \textbf{ holds } \mathrm{rng}\, f=Y, \tag{24}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ st } Y=\emptyset \textbf{ implies } X=\emptyset \textbf{ holds } f \text{ is\_one-to-one} \textbf{ iff for } x1,x2 \textbf{ st } x1\in X \;\&\; x2\in X \;\&\; f.x1=f.x2 \textbf{ holds } x1=x2, \tag{25}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y,Z \textbf{ st } (Z=\emptyset \textbf{ implies } Y=\emptyset) \;\&\; (Y=\emptyset \textbf{ implies } X=\emptyset) \;\&\; g\cdot f \text{ is\_one-to-one} \textbf{ holds } f \text{ is\_one-to-one}, \tag{26}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ st } X \neq \emptyset \mathbin{\&} Y \neq \emptyset \textbf{ holds } f \text{ is\_one-to-one} \\ \textbf{iff for } Z \textbf{ for } g, h \textbf{ being } \mathrm{Function} \textbf{ of } Z, X \textbf{ st } f \cdot g = f \cdot h \textbf{ holds } g = h, \tag{27}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y, Z \\ \textbf{st } Z \neq \emptyset \mathbin{\&} Y \neq \emptyset \mathbin{\&} \mathrm{rng}\,(g \cdot f) = Z \mathbin{\&} g \text{ is\_one-to-one} \textbf{ holds } \mathrm{rng}\, f = Y, \tag{28}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y, X \\ \textbf{st } X \neq \emptyset \mathbin{\&} Y \neq \emptyset \mathbin{\&} g \cdot f = \mathrm{id}\, X \textbf{ holds } f \text{ is\_one-to-one} \mathbin{\&} \mathrm{rng}\, g = X, \tag{29}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y, Z \textbf{ st} \\ (Z = \emptyset \textbf{ implies } Y = \emptyset) \mathbin{\&} g \cdot f \text{ is\_one-to-one} \mathbin{\&} \mathrm{rng}\, f = Y \\ \textbf{holds } f \text{ is\_one-to-one} \mathbin{\&} g \text{ is\_one-to-one}, \tag{30}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ st} \\ f \text{ is\_one-to-one} \mathbin{\&} (X = \emptyset \textbf{ iff } Y = \emptyset) \mathbin{\&} \mathrm{rng}\, f = Y \\ \textbf{holds } f^{-1} \textbf{ is } \mathrm{Function} \textbf{ of } Y, X, \tag{31}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \\ \textbf{st } Y \neq \emptyset \mathbin{\&} f \text{ is\_one-to-one} \mathbin{\&} x \in X \textbf{ holds } (f^{-1}).(f.x) = x, \tag{32}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \\ \textbf{st } \mathrm{rng}\, f = Y \mathbin{\&} f \text{ is\_one-to-one} \mathbin{\&} y \in Y \textbf{ holds } f.((f^{-1}).y) = y, \tag{33}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y, X \textbf{ st} \\ X \neq \emptyset \mathbin{\&} Y \neq \emptyset \mathbin{\&} \mathrm{rng}\, f = Y \\ \mathbin{\&} f \text{ is\_one-to-one} \mathbin{\&} \textbf{for } y, x \textbf{ holds } y \in Y \mathbin{\&} g.y = x \textbf{ iff } x \in X \mathbin{\&} f.x = y \\ \textbf{holds } g = f^{-1}, \tag{34}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \\ \textbf{st } Y \neq \emptyset \mathbin{\&} \mathrm{rng}\, f = Y \mathbin{\&} f \text{ is\_one-to-one} \textbf{ holds } f^{-1} \cdot f = \mathrm{id}\, X \mathbin{\&} f \cdot f^{-1} = \mathrm{id}\, Y, \tag{35}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y, X \textbf{ st} \\ X \neq \emptyset \mathbin{\&} Y \neq \emptyset \mathbin{\&} \mathrm{rng}\, f = Y \mathbin{\&} g \cdot f = \mathrm{id}\, X \mathbin{\&} f \text{ is\_one-to-one} \textbf{ holds } g = f^{-1}, \tag{36}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ st} \\ Y \neq \emptyset \mathbin{\&} \textbf{ex } g \textbf{ being } \mathrm{Function} \textbf{ of } Y, X \textbf{ st } g \cdot f = \mathrm{id}\, X \textbf{ holds } f \text{ is\_one-to-one}, \tag{37}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \\ \textbf{st } (Y = \emptyset \textbf{ implies } X = \emptyset) \mathbin{\&} Z \subseteq X \textbf{ holds } f \mid Z \textbf{ is } \mathrm{Function} \textbf{ of } Z, Y, \tag{38}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \\ \textbf{st } Y \neq \emptyset \mathbin{\&} x \in X \mathbin{\&} x \in Z \textbf{ holds } (f \mid Z).x = f.x, \tag{39}$$

(40) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y$$
$$\textbf{st } (Y = \emptyset \textbf{ implies } X = \emptyset) \ \& \ X \subseteq Z \textbf{ holds } f \mid Z = f,$$

(41) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y$$
$$\textbf{st } Y \neq \emptyset \ \& \ x \in X \ \& \ f.x \in Z \textbf{ holds } (Z \mid f).x = f.x,$$

(42) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y$$
$$\textbf{st } (Y = \emptyset \textbf{ implies } X = \emptyset) \ \& \ Y \subseteq Z \textbf{ holds } Z \mid f = f,$$

(43) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y$$
$$\textbf{st } Y \neq \emptyset \textbf{ for } y \textbf{ holds } y \in f^{\circ} P \textbf{ iff } \textbf{ex } x \textbf{ st } x \in X \ \& \ x \in P \ \& \ y = f.x,$$

(44) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ st } Y = \emptyset \textbf{ implies } X = \emptyset \textbf{ holds } f^{\circ} P \subseteq Y,$$

(45) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ st } Y = \emptyset \textbf{ implies } X = \emptyset \textbf{ holds } f^{\circ} X = \operatorname{rng} f,$$

(46) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y$$
$$\textbf{st } Y \neq \emptyset \textbf{ for } x \textbf{ holds } x \in f^{-1} Q \textbf{ iff } x \in X \ \& \ f.x \in Q,$$

(47) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ st } Y = \emptyset \textbf{ implies } X = \emptyset \textbf{ holds } f^{-1} Q \subseteq X,$$

(48) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ st } Y = \emptyset \textbf{ implies } X = \emptyset \textbf{ holds } f^{-1} Y = X,$$

(49) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y$$
$$\textbf{st } Y \neq \emptyset \textbf{ holds } (\textbf{for } y \textbf{ st } y \in Y \textbf{ holds } f^{-1} \{y\} \neq \emptyset) \textbf{ iff } \operatorname{rng} f = Y,$$

(50) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y$$
$$\textbf{st } (Y = \emptyset \textbf{ implies } X = \emptyset) \ \& \ P \subseteq X \textbf{ holds } P \subseteq f^{-1} (f^{\circ} P),$$

(51) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y$$
$$\textbf{st } Y = \emptyset \textbf{ implies } X = \emptyset \textbf{ holds } f^{-1} (f^{\circ} X) = X,$$

(52) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y$$
$$\textbf{st } (Y = \emptyset \textbf{ implies } X = \emptyset) \ \& \ \operatorname{rng} f = Y \textbf{ holds } f^{\circ} (f^{-1} Y) = Y,$$

(53) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } Y, Z \textbf{ st}$$
$$(Z = \emptyset \textbf{ implies } Y = \emptyset) \ \& \ (Y = \emptyset \textbf{ implies } X = \emptyset)$$
$$\textbf{holds } f^{-1} Q \subseteq (g \cdot f)^{-1} (g^{\circ} Q),$$

(54) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \emptyset, Y \textbf{ holds } \operatorname{dom} f = \emptyset \ \& \ \operatorname{rng} f = \emptyset,$$

(55) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{st } \operatorname{dom} f = \emptyset \textbf{ holds } f \textbf{ is } \text{Function } \textbf{of } \emptyset, Y,$$

(56) $$\textbf{for } f1 \textbf{ being } \text{Function } \textbf{of } \emptyset, Y1 \textbf{ for } f2 \textbf{ being } \text{Function } \textbf{of } \emptyset, Y2 \textbf{ holds } f1 = f2,$$

(57) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \emptyset, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } Y, Z \textbf{ st } Z = \emptyset \textbf{ implies } Y = \emptyset \textbf{ holds } g \cdot f \textbf{ is } \text{Function } \textbf{of } \emptyset, Z,$$

(58) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \emptyset, Y \textbf{ holds } f \text{ is\_one-to-one},$$

(59) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \emptyset, Y \textbf{ holds } f \circ P = \emptyset,$$

(60) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \emptyset, Y \textbf{ holds } f^{-1} Q = \emptyset,$$

(61) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \{x\}, Y \textbf{ st } Y \neq \emptyset \textbf{ holds } f.x \in Y,$$

(62) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \{x\}, Y \textbf{ st } Y \neq \emptyset \textbf{ holds } \text{rng } f = \{f.x\},$$

(63) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \{x\}, Y \textbf{ st } Y \neq \emptyset \textbf{ holds } f \text{ is\_one-to-one},$$

(64) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \{x\}, Y \textbf{ st } Y \neq \emptyset \textbf{ holds } f \circ P \subseteq \{f.x\},$$

(65) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, \{y\} \textbf{ st } x \in X \textbf{ holds } f.x = y,$$

(66) $$\textbf{for } f1, f2 \textbf{ being } \text{Function } \textbf{of } X, \{y\} \textbf{ holds } f1 = f2.$$

The arguments of the notions defined below are the following: X which is an object of the type reserved above; f, g which are objects of the type Function **of** X, X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$g \cdot f \quad \text{is} \quad \text{Function } \textbf{of } X, X.$$

Let us consider X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\text{id } X \quad \text{is} \quad \text{Function } \textbf{of } X, X.$$

The following propositions are true:

(67) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } \text{dom } f = X \ \& \ \text{rng } f \subseteq X,$$

(68) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{st } \text{dom } f = X \ \& \ \text{rng } f \subseteq X \textbf{ holds } f \textbf{ is } \text{Function } \textbf{of } X, X,$$

(69) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ st } x \in X \textbf{ holds } f.x \in X,$$

(70) $$\textbf{for } f, g \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ st } x \in X \textbf{ holds } (g \cdot f).x = g.(f.x),$$

(71) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ st } Y \neq \emptyset \ \& \ x \in X \textbf{ holds } (g \cdot f).x = g.(f.x),$$

(72) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } Y, Y \textbf{ st } Y \neq \emptyset \ \& \ x \in X \textbf{ holds } (g \cdot f).x = g.(f.x),$$

(73) $$\textbf{for } f,g \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ st } \operatorname{rng} f = X \ \& \ \operatorname{rng} g = X \textbf{ holds } \operatorname{rng}(g \cdot f) = X,$$

(74) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } f \cdot (\operatorname{id} X) = f \ \& \ (\operatorname{id} X) \cdot f = f,$$

(75) $$\textbf{for } f,g \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ st } g \cdot f = f \ \& \ \operatorname{rng} f = X \textbf{ holds } g = \operatorname{id} X,$$

(76) $$\textbf{for } f,g \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ st } f \cdot g = f \ \& \ f \text{ is\_one-to-one } \textbf{holds } g = \operatorname{id} X,$$

(77) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } f \text{ is\_one-to-one } \textbf{iff for } x1,x2 \textbf{ st } x1 \in X \ \& \ x2 \in X \ \& \ f.x1 = f.x2 \textbf{ holds } x1 = x2,$$

(78) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } f^{\circ} P \subseteq X.$$

The arguments of the notions defined below are the following: X which is an object of the type reserved above; f which is an object of the type Function **of** X, X; P which is an object of the type reserved above. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f^{\circ} P \qquad \text{is} \qquad \text{Subset } \textbf{of } X.$$

One can prove the following propositions:

(79) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } f^{\circ} X = \operatorname{rng} f,$$

(80) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } f^{-1} Q \subseteq X.$$

The arguments of the notions defined below are the following: X which is an object of the type reserved above; f which is an object of the type Function **of** X, X; Q which is an object of the type reserved above. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f^{-1} Q \qquad \text{is} \qquad \text{Subset } \textbf{of } X.$$

Next we state two propositions:

(81) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ st } \operatorname{rng} f = X \textbf{ holds } f^{\circ} (f^{-1} X) = X,$$

(82) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } f^{-1} (f^{\circ} X) = X.$$

Let us consider X. The mode

$$\text{Permutation } \textbf{of } X,$$

which widens to the type Function **of** X, X, is defined by

$$\textbf{it} \text{ is\_one-to-one } \& \operatorname{rng} \textbf{it} = X.$$

Next we state three propositions:

(83) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X$$
$$\textbf{holds } f \textbf{ is } \text{Permutation } \textbf{of } X \textbf{ iff } f \text{ is\_one-to-one} \ \& \ \text{rng}\, f = X,$$

(84) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ holds } f \text{ is\_one-to-one} \ \& \ \text{rng}\, f = X,$$

(85) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X$$
$$\textbf{for } x1,x2 \textbf{ st } x1 \in X \ \& \ x2 \in X \ \& \ f\,.x1 = f\,.x2 \textbf{ holds } x1 = x2.$$

The arguments of the notions defined below are the following: X which is an object of the type reserved above; f, g which are objects of the type Permutation **of** X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$g \cdot f \qquad \text{is} \qquad \text{Permutation } \textbf{of } X.$$

Let us consider X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\text{id}\, X \qquad \text{is} \qquad \text{Permutation } \textbf{of } X.$$

The arguments of the notions defined below are the following: X which is an object of the type reserved above; f which is an object of the type Permutation **of** X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f^{-1} \qquad \text{is} \qquad \text{Permutation } \textbf{of } X.$$

The following propositions are true:

(86) $$\textbf{for } f,g \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ st } g \cdot f = g \textbf{ holds } f = \text{id}\, X,$$

(87) $$\textbf{for } f,g \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ st } g \cdot f = \text{id}\, X \textbf{ holds } g = f^{-1},$$

(88) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ holds } (f^{-1}) \cdot f = \text{id}\, X \ \& \ f \cdot (f^{-1}) = \text{id}\, X,$$

(89) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ holds } (f^{-1})^{-1} = f,$$

(90) $$\textbf{for } f,g \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ holds } (g \cdot f)^{-1} = f^{-1} \cdot g^{-1},$$

(91) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ st } P \cap Q = \emptyset \textbf{ holds } f^{\circ}\, P \cap f^{\circ}\, Q = \emptyset,$$

(92) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X$$
$$\textbf{st } P \subseteq X \textbf{ holds } f^{\circ}\,(f^{-1}\, P) = P \ \& \ f^{-1}\,(f^{\circ}\, P) = P,$$

(93) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ holds } f^{\circ}\, P = (f^{-1})^{-1}\, P \ \& \ f^{-1}\, P = (f^{-1})^{\circ}\, P.$$

In the sequel C, D, E denote objects of the type DOMAIN. The arguments of the notions defined below are the following: X, D, E which are objects of the type reserved above; f which is an object of the type Function **of** X, D; g which is an object of the type Function **of** D, E. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$g \cdot f \qquad \text{is} \qquad \text{Function } \textbf{of } X, E.$$

Let us consider X, D. Let us note that one can characterize the mode

$$\text{Function } \textbf{of } X, D$$

by the following (equivalent) condition:

$$X = \text{dom}\,\textbf{it} \ \& \ \text{rng}\,\textbf{it} \subseteq D.$$

We now state a number of propositions:

(94) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } \text{dom}\, f = X \ \& \ \text{rng}\, f \subseteq D,$$

(95) $$\textbf{for } f \textbf{ being } \text{Function} \\ \textbf{st } \text{dom}\, f = X \ \& \ \text{rng}\, f \subseteq D \textbf{ holds } f \textbf{ is } \text{Function } \textbf{of } X, D,$$

(96) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ st } x \in X \textbf{ holds } f.x \in D,$$

(97) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \{x\}, D \textbf{ holds } f.x \in D,$$

(98) $$\textbf{for } f1, f2 \textbf{ being } \text{Function } \textbf{of } X, D \\ \textbf{st } \textbf{for } x \textbf{ st } x \in X \textbf{ holds } f1.x = f2.x \textbf{ holds } f1 = f2,$$

(99) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \\ \textbf{for } g \textbf{ being } \text{Function } \textbf{of } D, E \textbf{ st } x \in X \textbf{ holds } (g \cdot f).x = g.(f.x),$$

(100) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } f \cdot (\text{id}\, X) = f \ \& \ (\text{id}\, D) \cdot f = f,$$

(101) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } f \text{ is\_one-to-one} \\ \textbf{iff } \textbf{for } x1, x2 \textbf{ st } x1 \in X \ \& \ x2 \in X \ \& \ f.x1 = f.x2 \textbf{ holds } x1 = x2,$$

(102) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \\ \textbf{for } y \textbf{ holds } y \in f \circ P \textbf{ iff } \textbf{ex } x \textbf{ st } x \in X \ \& \ x \in P \ \& \ y = f.x,$$

(103) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } f \circ P \subseteq D.$$

The arguments of the notions defined below are the following: X, D which are objects of the type reserved above; f which is an object of the type Function **of** X, D; P which is an object of the type reserved above. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f \circ P \qquad \text{is} \qquad \text{Subset } \textbf{of } D.$$

One can prove the following propositions:

(104) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } f^{\circ} X = \operatorname{rng} f,$$

(105) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ st } f^{\circ} X = D \textbf{ holds } \operatorname{rng}(f) = D,$$

(106) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ for } x \textbf{ holds } x \in f^{-1} Q \textbf{ iff } x \in X \;\&\; f.x \in Q,$$

(107) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } f^{-1} Q \subseteq X.$$

The arguments of the notions defined below are the following: X, D which are objects of the type reserved above; f which is an object of the type Function **of** X, D; Q which is an object of the type reserved above. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f^{-1} Q \quad \text{is} \quad \text{Subset } \textbf{of } X.$$

One can prove the following propositions:

(108) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } f^{-1} D = X,$$

(109) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D$$
$$\textbf{holds } (\textbf{for } y \textbf{ st } y \in D \textbf{ holds } f^{-1} \{y\} \neq \emptyset) \textbf{ iff } \operatorname{rng} f = D,$$

(110) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } f^{-1} (f^{\circ} X) = X,$$

(111) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ st } \operatorname{rng} f = D \textbf{ holds } f^{\circ} (f^{-1} D) = D,$$

(112) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D$$
$$\textbf{for } g \textbf{ being } \text{Function } \textbf{of } D, E \textbf{ holds } f^{-1} Q \subseteq (g \cdot f)^{-1} (g^{\circ} Q).$$

In the sequel c denotes an object of the type Element **of** C; d denotes an object of the type Element **of** D. The arguments of the notions defined below are the following: C, D which are objects of the type reserved above; f which is an object of the type Function **of** C, D; c which is an object of the type reserved above. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f.c \quad \text{is} \quad \text{Element } \textbf{of } D.$$

Now we present two schemes. The scheme *FuncExD* concerns a constant $\mathcal{A}$ that has the type DOMAIN, a constant $\mathcal{B}$ that has the type DOMAIN and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } f \textbf{ being } \text{Function } \textbf{of } \mathcal{A}, \mathcal{B} \textbf{ st for } x \textbf{ being } \text{Element } \textbf{of } \mathcal{A} \textbf{ holds } \mathcal{P}[x, f.x]$$

provided the parameters satisfy the following conditions:

- $\textbf{for } x \textbf{ being } \text{Element } \textbf{of } \mathcal{A} \textbf{ ex } y \textbf{ being } \text{Element } \textbf{of } \mathcal{B} \textbf{ st } \mathcal{P}[x, y],$

- $$\textbf{for}\ x\ \textbf{being}\ \text{Element}\ \textbf{of}\ \mathcal{A}, y1,y2\ \textbf{being}\ \text{Element}\ \textbf{of}\ \mathcal{B}\ \textbf{st}\ \mathcal{P}[x,y1]\ \&\ \mathcal{P}[x,y2]\ \textbf{holds}\ y1 = y2.$$

The scheme *LambdaD* concerns a constant $\mathcal{A}$ that has the type DOMAIN, a constant $\mathcal{B}$ that has the type DOMAIN and a unary functor $\mathcal{F}$ yielding values of the type Element **of** $\mathcal{B}$ and states that the following holds

$$\textbf{ex}\ f\ \textbf{being}\ \text{Function}\ \textbf{of}\ \mathcal{A},\mathcal{B}\ \textbf{st}\ \textbf{for}\ x\ \textbf{being}\ \text{Element}\ \textbf{of}\ \mathcal{A}\ \textbf{holds}\ f.x = \mathcal{F}(x)$$

for all values of the parameters.

One can prove the following propositions:

(113) $\textbf{for}\ f1,f2\ \textbf{being}\ \text{Function}\ \textbf{of}\ C,D\ \textbf{st}\ \textbf{for}\ c\ \textbf{holds}\ f1.c = f2.c\ \textbf{holds}\ f1 = f2,$

(114) $$(\text{id}\, C).c = c,$$

(115) $$\textbf{for}\ f\ \textbf{being}\ \text{Function}\ \textbf{of}\ C,D\ \textbf{for}\ g\ \textbf{being}\ \text{Function}\ \textbf{of}\ D,E\ \textbf{holds}\ (g \cdot f).c = g.(f.c),$$

(116) $$\textbf{for}\ f\ \textbf{being}\ \text{Function}\ \textbf{of}\ C,D\ \textbf{for}\ d\ \textbf{holds}\ d \in f\ ^{\circ}\ P\ \textbf{iff}\ \textbf{ex}\ c\ \textbf{st}\ c \in P\ \&\ d = f.c,$$

(117) $$\textbf{for}\ f\ \textbf{being}\ \text{Function}\ \textbf{of}\ C,D\ \textbf{for}\ c\ \textbf{holds}\ c \in f^{-1}\ Q\ \textbf{iff}\ f.c \in Q,$$

(118) $$\textbf{for}\ f1,f2\ \textbf{being}\ \text{Function}\ \textbf{of}\ [:X,Y:],Z\ \textbf{st}\ Z \neq \emptyset\ \&\ \textbf{for}\ x,y\ \textbf{st}\ x \in X\ \&\ y \in Y\ \textbf{holds}\ f1.\langle x,y\rangle = f2.\langle x,y\rangle\ \textbf{holds}\ f1 = f2,$$

(119) $$\textbf{for}\ f\ \textbf{being}\ \text{Function}\ \textbf{of}\ [:X,Y:],Z\ \textbf{st}\ x \in X\ \&\ y \in Y\ \&\ Z \neq \emptyset\ \textbf{holds}\ f.\langle x,y\rangle \in Z.$$

Now we present two schemes. The scheme *FuncEx2* concerns a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set, a constant $\mathcal{C}$ that has the type set and a ternary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex}\ f\ \textbf{being}\ \text{Function}\ \textbf{of}\ [:\mathcal{A},\mathcal{B}:],\mathcal{C}\ \textbf{st}\ \textbf{for}\ x,y\ \textbf{st}\ x \in \mathcal{A}\ \&\ y \in \mathcal{B}\ \textbf{holds}\ \mathcal{P}[x,y,f.\langle x,y\rangle]$$

provided the parameters satisfy the following conditions:

- $$\textbf{for}\ x,y\ \textbf{st}\ x \in \mathcal{A}\ \&\ y \in \mathcal{B}\ \textbf{ex}\ z\ \textbf{st}\ z \in \mathcal{C}\ \&\ \mathcal{P}[x,y,z],$$
- $$\textbf{for}\ x,y,z1,z2\ \textbf{st}\ x \in \mathcal{A}\ \&\ y \in \mathcal{B}\ \&\ \mathcal{P}[x,y,z1]\ \&\ \mathcal{P}[x,y,z2]\ \textbf{holds}\ z1 = z2.$$

The scheme *Lambda2* concerns a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set, a constant $\mathcal{C}$ that has the type set and a binary functor $\mathcal{F}$ and states that the following holds

$$\textbf{ex}\ f\ \textbf{being}\ \text{Function}\ \textbf{of}\ [:\mathcal{A},\mathcal{B}:],\mathcal{C}\ \textbf{st}\ \textbf{for}\ x,y\ \textbf{st}\ x \in \mathcal{A}\ \&\ y \in \mathcal{B}\ \textbf{holds}\ f.\langle x,y\rangle = \mathcal{F}(x,y)$$

provided the parameters satisfy the following condition:

- **for** x,y **st** $x \in \mathcal{A}$ & $y \in \mathcal{B}$ **holds** $\mathcal{F}(x, y) \in \mathcal{C}$.

We now state a proposition

(120) **for** $f1$,$f2$ **being** Function **of** [:C, D:],E
st for c,d **holds** $f1.\langle c, d\rangle = f2.\langle c, d\rangle$ **holds** $f1 = f2$.

Now we present two schemes. The scheme *FuncEx2D* deals with a constant $\mathcal{A}$ that has the type DOMAIN, a constant $\mathcal{B}$ that has the type DOMAIN, a constant $\mathcal{C}$ that has the type DOMAIN and a ternary predicate $\mathcal{P}$ and states that the following holds

ex f **being** Function **of** [:$\mathcal{A}$,$\mathcal{B}$:],$\mathcal{C}$
st for x **being** Element **of** $\mathcal{A}$ **for** y **being** Element **of** $\mathcal{B}$ **holds** $\mathcal{P}[x, y, f.\langle x, y\rangle]$

provided the parameters satisfy the following conditions:

- **for** x **being** Element **of** $\mathcal{A}$
for y **being** Element **of** $\mathcal{B}$ **ex** z **being** Element **of** $\mathcal{C}$ **st** $\mathcal{P}[x, y, z]$,
- **for** x **being** Element **of** $\mathcal{A}$ **for** y **being** Element **of** $\mathcal{B}$
for $z1$,$z2$ **being** Element **of** $\mathcal{C}$ **st** $\mathcal{P}[x, y, z1]$ & $\mathcal{P}[x, y, z2]$ **holds** $z1 = z2$.

The scheme *Lambda2D* concerns a constant $\mathcal{A}$ that has the type DOMAIN, a constant $\mathcal{B}$ that has the type DOMAIN, a constant $\mathcal{C}$ that has the type DOMAIN and a binary functor $\mathcal{F}$ yielding values of the type Element **of** $\mathcal{C}$ and states that the following holds

ex f **being** Function **of** [:$\mathcal{A}$,$\mathcal{B}$:],$\mathcal{C}$
st for x **being** Element **of** $\mathcal{A}$ **for** y **being** Element **of** $\mathcal{B}$ **holds** $f.\langle x, y\rangle = \mathcal{F}(x, y)$

for all values of the parameters.

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Received April 6, 1989