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Tuples, Projections and Cartesian Products

Andrzej Trybulec[1]
Warsaw University
Białystok

Summary. The purpose of this article is to define projections of ordered pairs, and to introduce triples and quadruples, and their projections. The theorems in this paper may be roughly divided into two groups: theorems describing basic properties of introduced concepts and theorems related to the regularity, analogous to those proved for ordered pairs by Cz. Byliński [1]. Cartesian products of subsets are redefined as subsets of Cartesian products.

The notation and terminology used here are introduced in the following papers: [3], [4], and [2]. For simplicity we adopt the following convention: v, x, $x1$, $x2$, $x3$, $x4$, y, $y1$, $y2$, $y3$, $y4$, z denote objects of the type Any; X, $X1$, $X2$, $X3$, $X4$, Y, $Y1$, $Y2$, $Y3$, $Y4$, $Y5$, Z denote objects of the type set. One can prove the following propositions:

$$X \neq \emptyset \ \mathbf{implies} \ \mathbf{ex}\, Y \ \mathbf{st} \ Y \in X \ \& \ Y \text{ misses } X, \tag{1}$$

$$X \neq \emptyset \ \mathbf{implies} \ \mathbf{ex}\, Y \ \mathbf{st} \ Y \in X \ \& \ \mathbf{for}\, Y1 \ \mathbf{st} \ Y1 \in Y \ \mathbf{holds} \ Y1 \text{ misses } X, \tag{2}$$

$$X \neq \emptyset \ \mathbf{implies} \\ \mathbf{ex}\, Y \ \mathbf{st} \ Y \in X \ \& \ \mathbf{for}\, Y1{,}Y2 \ \mathbf{st} \ Y1 \in Y2 \ \& \ Y2 \in Y \ \mathbf{holds} \ Y1 \text{ misses } X, \tag{3}$$

$$X \neq \emptyset \ \mathbf{implies} \ \mathbf{ex}\, Y \ \mathbf{st} \ Y \in X \\ \& \ \mathbf{for}\, Y1{,}Y2{,}Y3 \ \mathbf{st} \ Y1 \in Y2 \ \& \ Y2 \in Y3 \ \& \ Y3 \in Y \ \mathbf{holds} \ Y1 \text{ misses } X, \tag{4}$$

$$X \neq \emptyset \ \mathbf{implies} \ \mathbf{ex}\, Y \ \mathbf{st} \ Y \in X \ \& \ \mathbf{for}\, Y1{,}Y2{,}Y3{,}Y4 \\ \mathbf{st} \ Y1 \in Y2 \ \& \ Y2 \in Y3 \ \& \ Y3 \in Y4 \ \& \ Y4 \in Y \ \mathbf{holds} \ Y1 \text{ misses } X, \tag{5}$$

$$X \neq \emptyset \ \mathbf{implies} \ \mathbf{ex}\, Y \ \mathbf{st} \ Y \in X \ \& \ \mathbf{for}\, Y1{,}Y2{,}Y3{,}Y4{,}Y5 \ \mathbf{st} \\ Y1 \in Y2 \ \& \ Y2 \in Y3 \ \& \ Y3 \in Y4 \ \& \ Y4 \in Y5 \ \& \ Y5 \in Y \ \mathbf{holds} \ Y1 \text{ misses } X. \tag{6}$$

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We now define two new functors. Let us consider x. Assume there exist $x1$, $x2$, of the type Any such that

$$x = \langle x1{,}x2\rangle.$$

The functor

$$x_{\mathbf{1}}\,,$$

is defined by

$$x = \langle y1{,}y2\rangle \textbf{ implies it} = y1.$$

The functor

$$x_{\mathbf{2}}\,,$$

is defined by

$$x = \langle y1{,}y2\rangle \textbf{ implies it} = y2.$$

We now state a number of propositions:

$$\langle x, y\rangle_{\mathbf{1}} = x \ \&\ \langle x, y\rangle_{\mathbf{2}} = y, \tag{7}$$

$$(\mathbf{ex}\, x{,}y \textbf{ st } z = \langle x, y\rangle) \textbf{ implies } \langle z_{\mathbf{1}}, z_{\mathbf{2}}\rangle = z, \tag{8}$$

$$X \neq \emptyset \textbf{ implies ex}\, v \textbf{ st } v \in X \ \&\ \textbf{not ex}\, x{,}y \textbf{ st } (x \in X \textbf{ or } y \in X) \ \&\ v = \langle x, y\rangle, \tag{9}$$

$$z \in [\!:X, Y:\!] \textbf{ implies } z_{\mathbf{1}} \in X \ \&\ z_{\mathbf{2}} \in Y, \tag{10}$$

$$(\mathbf{ex}\, x{,}y \textbf{ st } z = \langle x, y\rangle) \ \&\ z_{\mathbf{1}} \in X \ \&\ z_{\mathbf{2}} \in Y \textbf{ implies } z \in [\!:X, Y:\!], \tag{11}$$

$$z \in [\!:\{x\}{,}Y:\!] \textbf{ implies } z_{\mathbf{1}} = x \ \&\ z_{\mathbf{2}} \in Y, \tag{12}$$

$$z \in [\!:X, \{y\}:\!] \textbf{ implies } z_{\mathbf{1}} \in X \ \&\ z_{\mathbf{2}} = y, \tag{13}$$

$$z \in [\!:\{x\}{,}\{y\}:\!] \textbf{ implies } z_{\mathbf{1}} = x \ \&\ z_{\mathbf{2}} = y, \tag{14}$$

$$z \in [\!:\{x1{,}x2\}{,}Y:\!] \textbf{ implies } (z_{\mathbf{1}} = x1 \textbf{ or } z_{\mathbf{1}} = x2) \ \&\ z_{\mathbf{2}} \in Y, \tag{15}$$

$$z \in [\!:X, \{y1{,}y2\}:\!] \textbf{ implies } z_{\mathbf{1}} \in X \ \&\ (z_{\mathbf{2}} = y1 \textbf{ or } z_{\mathbf{2}} = y2), \tag{16}$$

$$z \in [\!:\{x1{,}x2\}{,}\{y\}:\!] \textbf{ implies } (z_{\mathbf{1}} = x1 \textbf{ or } z_{\mathbf{1}} = x2) \ \&\ z_{\mathbf{2}} = y, \tag{17}$$

$$z \in [\!:\{x\}{,}\{y1{,}y2\}:\!] \textbf{ implies } z_{\mathbf{1}} = x \ \&\ (z_{\mathbf{2}} = y1 \textbf{ or } z_{\mathbf{2}} = y2), \tag{18}$$

$$\begin{gathered} z \in [\!:\{x1{,}x2\}{,}\{y1{,}y2\}:\!] \\ \textbf{implies } (z_{\mathbf{1}} = x1 \textbf{ or } z_{\mathbf{1}} = x2) \ \&\ (z_{\mathbf{2}} = y1 \textbf{ or } z_{\mathbf{2}} = y2), \end{gathered} \tag{19}$$

$$(\mathbf{ex}\, y{,}z \textbf{ st } x = \langle y, z\rangle) \textbf{ implies } x \neq x_{\mathbf{1}} \ \&\ x \neq x_{\mathbf{2}}\,. \tag{20}$$

In the sequel xx will have the type Element **of** X; yy will have the type Element **of** Y. One can prove the following propositions:

(21) $$X \neq \emptyset \ \& \ Y \neq \emptyset \ \textbf{implies} \ \langle xx, yy \rangle \in [:X, Y:],$$

(22) $$X \neq \emptyset \ \& \ Y \neq \emptyset \ \textbf{implies} \ \langle xx, yy \rangle \ \textbf{is} \ \text{Element} \ \textbf{of} \ [:X, Y:],$$

(23) $$x \in [:X, Y:] \ \textbf{implies} \ x = \langle x_{\mathbf{1}}, x_{\mathbf{2}} \rangle,$$

(24) $$X \neq \emptyset \ \& \ Y \neq \emptyset \ \textbf{implies for} \ x \ \textbf{being} \ \text{Element} \ \textbf{of} \ [:X, Y:] \ \textbf{holds} \ x = \langle x_{\mathbf{1}}, x_{\mathbf{2}} \rangle,$$

(25) $$[:\{x1, x2\}, \{y1, y2\}:] = \{\langle x1, y1 \rangle, \langle x1, y2 \rangle, \langle x2, y1 \rangle, \langle x2, y2 \rangle\},$$

(26) $$X \neq \emptyset \ \& \ Y \neq \emptyset$$
$$\textbf{implies for} \ x \ \textbf{being} \ \text{Element} \ \textbf{of} \ [:X, Y:] \ \textbf{holds} \ x \neq x_{\mathbf{1}} \ \& \ x \neq x_{\mathbf{2}} \, .$$

Let us consider $x1$, $x2$, $x3$. The functor

$$\langle x1, x2, x3 \rangle,$$

is defined by

$$\textbf{it} = \langle \langle x1, x2 \rangle, x3 \rangle.$$

One can prove the following three propositions:

(27) $$\langle x1, x2, x3 \rangle = \langle \langle x1, x2 \rangle, x3 \rangle,$$

(28) $$\langle x1, x2, x3 \rangle = \langle y1, y2, y3 \rangle \ \textbf{implies} \ x1 = y1 \ \& \ x2 = y2 \ \& \ x3 = y3,$$

(29) $$X \neq \emptyset$$
$$\textbf{implies ex} \ v \ \textbf{st} \ v \in X \ \& \ \textbf{not ex} \ x, y, z \ \textbf{st} \ (x \in X \ \textbf{or} \ y \in X) \ \& \ v = \langle x, y, z \rangle.$$

Let us consider $x1$, $x2$, $x3$, $x4$. The functor

$$\langle x1, x2, x3, x4 \rangle,$$

is defined by

$$\textbf{it} = \langle \langle x1, x2, x3 \rangle, x4 \rangle.$$

The following propositions are true:

(30) $$\langle x1, x2, x3, x4 \rangle = \langle \langle x1, x2, x3 \rangle, x4 \rangle,$$

(31) $$\langle x1, x2, x3, x4 \rangle = \langle \langle \langle x1, x2 \rangle, x3 \rangle, x4 \rangle,$$

(32) $$\langle x1, x2, x3, x4 \rangle = \langle \langle x1, x2 \rangle, x3, x4 \rangle,$$

(33) $$\langle x1, x2, x3, x4 \rangle = \langle y1, y2, y3, y4 \rangle$$
$$\textbf{implies} \ x1 = y1 \ \& \ x2 = y2 \ \& \ x3 = y3 \ \& \ x4 = y4,$$

$$X \neq \emptyset \textbf{ implies ex } v \textbf{ st } v \in X \ \& \ \textbf{not ex } x1,x2,x3,x4 \textbf{ st } (x1 \in X \textbf{ or } x2 \in X) \ \& \ v = \langle x1,x2,x3,x4 \rangle, \tag{34}$$

$$X1 \neq \emptyset \ \& \ X2 \neq \emptyset \ \& \ X3 \neq \emptyset \textbf{ iff } [: X1,X2,X3 :] \neq \emptyset. \tag{35}$$

In the sequel $xx1$ has the type Element **of** $X1$; $xx2$ has the type Element **of** $X2$; $xx3$ has the type Element **of** $X3$. One can prove the following propositions:

$$X1 \neq \emptyset \ \& \ X2 \neq \emptyset \ \& \ X3 \neq \emptyset \textbf{ implies } ([: X1,X2,X3 :] = [: Y1,Y2,Y3 :] \textbf{ implies } X1 = Y1 \ \& \ X2 = Y2 \ \& \ X3 = Y3), \tag{36}$$

$$[: X1,X2,X3 :] \neq \emptyset \ \& \ [: X1,X2,X3 :] = [: Y1,Y2,Y3 :] \textbf{ implies } X1 = Y1 \ \& \ X2 = Y2 \ \& \ X3 = Y3, \tag{37}$$

$$[: X, X, X :] = [: Y, Y, Y :] \textbf{ implies } X = Y, \tag{38}$$

$$[: \{x1\},\{x2\},\{x3\} :] = \{\langle x1,x2,x3 \rangle\}, \tag{39}$$

$$[: \{x1,y1\},\{x2\},\{x3\} :] = \{\langle x1,x2,x3 \rangle,\langle y1,x2,x3 \rangle\}, \tag{40}$$

$$[: \{x1\},\{x2,y2\},\{x3\} :] = \{\langle x1,x2,x3 \rangle,\langle x1,y2,x3 \rangle\}, \tag{41}$$

$$[: \{x1\},\{x2\},\{x3,y3\} :] = \{\langle x1,x2,x3 \rangle,\langle x1,x2,y3 \rangle\}, \tag{42}$$

$$[: \{x1,y1\},\{x2,y2\},\{x3\} :] = \{\langle x1,x2,x3 \rangle,\langle y1,x2,x3 \rangle,\langle x1,y2,x3 \rangle,\langle y1,y2,x3 \rangle\}, \tag{43}$$

$$[: \{x1,y1\},\{x2\},\{x3,y3\} :] = \{\langle x1,x2,x3 \rangle,\langle y1,x2,x3 \rangle,\langle x1,x2,y3 \rangle,\langle y1,x2,y3 \rangle\}, \tag{44}$$

$$[: \{x1\},\{x2,y2\},\{x3,y3\} :] = \{\langle x1,x2,x3 \rangle,\langle x1,y2,x3 \rangle,\langle x1,x2,y3 \rangle,\langle x1,y2,y3 \rangle\}, \tag{45}$$

$$[: \{x1,y1\},\{x2,y2\},\{x3,y3\} :] = \{\langle x1,x2,x3 \rangle, \langle x1,y2,x3 \rangle,\langle x1,x2,y3 \rangle,\langle x1,y2,y3 \rangle,\langle y1,x2,x3 \rangle,\langle y1,y2,x3 \rangle,\langle y1,x2,y3 \rangle,\langle y1,y2,y3 \rangle\}. \tag{46}$$

We now define three new functors. Let us consider $X1$, $X2$, $X3$. Assume that the following holds

$$X1 \neq \emptyset \ \& \ X2 \neq \emptyset \ \& \ X3 \neq \emptyset.$$

Let x have the type Element **of** $[: X1,X2,X3 :]$. The functor

$$x_{\mathbf{1}},$$

with values of the type Element **of** $X1$, is defined by

$$x = \langle x1,x2,x3 \rangle \textbf{ implies it} = x1.$$

The functor

$$x_{\mathbf{2}},$$

yields the type Element **of** $X2$ and is defined by

$$x = \langle x1,x2,x3\rangle \textbf{ implies it} = x2.$$

The functor

$$x_{\mathbf{3}},$$

with values of the type Element **of** $X3$, is defined by

$$x = \langle x1,x2,x3\rangle \textbf{ implies it} = x3.$$

One can prove the following propositions:

(47) $X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset$ **implies for** x **being** Element **of** $[\!:X1,X2,X3:\!]$ **for** $x1,x2,x3$ **st** $x = \langle x1,x2,x3\rangle$ **holds** $x_{\mathbf{1}} = x1 \;\&\; x_{\mathbf{2}} = x2 \;\&\; x_{\mathbf{3}} = x3$,

(48) $X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset$ **implies for** x **being** Element **of** $[\!:X1,X2,X3:\!]$ **holds** $x = \langle x_{\mathbf{1}},x_{\mathbf{2}},x_{\mathbf{3}}\rangle$,

(49) $X \subseteq [\!:X,Y,Z:\!]$ **or** $X \subseteq [\!:Y,Z,X:\!]$ **or** $X \subseteq [\!:Z,X,Y:\!]$ **implies** $X = \emptyset$,

(50) $X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset$ **implies for** x **being** Element **of** $[\!:X1,X2,X3:\!]$ **holds** $x_{\mathbf{1}} = (x$ **qua** Any$)_{\mathbf{1}\,\mathbf{1}} \;\&\; x_{\mathbf{2}} = (x$ **qua** Any$)_{\mathbf{1}\,\mathbf{2}} \;\&\; x_{\mathbf{3}} = (x$ **qua** Any$)_{\mathbf{2}}$,

(51) $X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset$ **implies for** x **being** Element **of** $[\!:X1,X2,X3:\!]$ **holds** $x \neq x_{\mathbf{1}} \;\&\; x \neq x_{\mathbf{2}} \;\&\; x \neq x_{\mathbf{3}}$,

(52) $[\!:X1,X2,X3:\!]$ meets $[\!:Y1,Y2,Y3:\!]$ **implies** $X1$ meets $Y1$ & $X2$ meets $Y2$ & $X3$ meets $Y3$,

(53) $[\!:X1,X2,X3,X4:\!] = [\!:[\!:[\!:X1,X2:\!],X3:\!],X4:\!]$,

(54) $[\!:[\!:X1,X2:\!],X3,X4:\!] = [\!:X1,X2,X3,X4:\!]$,

(55) $X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset$ **iff** $[\!:X1,X2,X3,X4:\!] \neq \emptyset$,

(56) $X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset$ **implies** $([\!:X1,X2,X3,X4:\!] = [\!:Y1,Y2,Y3,Y4:\!]$ **implies** $X1 = Y1 \;\&\; X2 = Y2 \;\&\; X3 = Y3 \;\&\; X4 = Y4)$,

(57) $[\!:X1,X2,X3,X4:\!] \neq \emptyset \;\&\; [\!:X1,X2,X3,X4:\!] = [\!:Y1,Y2,Y3,Y4:\!]$ **implies** $X1 = Y1 \;\&\; X2 = Y2 \;\&\; X3 = Y3 \;\&\; X4 = Y4$,

(58) $[\!:X,X,X,X:\!] = [\!:Y,Y,Y,Y:\!]$ **implies** $X = Y$.

In the sequel $xx4$ will have the type Element **of** $X4$. We now define four new functors. Let us consider $X1$, $X2$, $X3$, $X4$. Assume that the following holds

$$X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset.$$

Let x have the type Element **of** $[\!: X1, X2, X3, X4 :\!]$. The functor

$$x_{\mathbf{1}},$$

yields the type Element **of** $X1$ and is defined by

$$x = \langle x1, x2, x3, x4 \rangle \textbf{ implies it} = x1.$$

The functor

$$x_{\mathbf{2}},$$

with values of the type Element **of** $X2$, is defined by

$$x = \langle x1, x2, x3, x4 \rangle \textbf{ implies it} = x2.$$

The functor

$$x_{\mathbf{3}},$$

yields the type Element **of** $X3$ and is defined by

$$x = \langle x1, x2, x3, x4 \rangle \textbf{ implies it} = x3.$$

The functor

$$x_{\mathbf{4}},$$

with values of the type Element **of** $X4$, is defined by

$$x = \langle x1, x2, x3, x4 \rangle \textbf{ implies it} = x4.$$

Next we state several propositions:

(59)
$$\begin{gathered} X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset \textbf{ implies} \\ \textbf{for } x \textbf{ being } \text{Element } \textbf{of } [\!: X1, X2, X3, X4 :\!] \textbf{ for } x1, x2, x3, x4 \\ \textbf{st } x = \langle x1, x2, x3, x4 \rangle \textbf{ holds } x_{\mathbf{1}} = x1 \;\&\; x_{\mathbf{2}} = x2 \;\&\; x_{\mathbf{3}} = x3 \;\&\; x_{\mathbf{4}} = x4, \end{gathered}$$

(60)
$$\begin{gathered} X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset \\ \textbf{implies for } x \textbf{ being } \text{Element } \textbf{of } [\!: X1, X2, X3, X4 :\!] \textbf{ holds } x = \langle x_{\mathbf{1}}, x_{\mathbf{2}}, x_{\mathbf{3}}, x_{\mathbf{4}} \rangle, \end{gathered}$$

(61)
$$\begin{gathered} X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset \textbf{ implies} \\ \textbf{for } x \textbf{ being } \text{Element } \textbf{of } [\!: X1, X2, X3, X4 :\!] \textbf{ holds } x_{\mathbf{1}} = (x \textbf{ qua } \text{Any})_{\mathbf{1}\,\mathbf{1}\,\mathbf{1}} \\ \&\; x_{\mathbf{2}} = (x \textbf{ qua } \text{Any})_{\mathbf{1}\,\mathbf{1}\,\mathbf{2}} \;\&\; x_{\mathbf{3}} = (x \textbf{ qua } \text{Any})_{\mathbf{1}\,\mathbf{2}} \;\&\; x_{\mathbf{4}} = (x \textbf{ qua } \text{Any})_{\mathbf{2}}, \end{gathered}$$

(62)
$$\begin{gathered} X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset \textbf{ implies} \\ \textbf{for } x \textbf{ being } \text{Element } \textbf{of } [\!: X1, X2, X3, X4 :\!] \\ \textbf{holds } x \neq x_{\mathbf{1}} \;\&\; x \neq x_{\mathbf{2}} \;\&\; x \neq x_{\mathbf{3}} \;\&\; x \neq x_{\mathbf{4}}, \end{gathered}$$

(63)
$$\begin{gathered} X1 \subseteq [\!: X1, X2, X3, X4 :\!] \textbf{ or} \\ X1 \subseteq [\!: X2, X3, X4, X1 :\!] \textbf{ or } X1 \subseteq [\!: X3, X4, X1, X2 :\!] \textbf{ or } X1 \subseteq [\!: X4, X1, X2, X3 :\!] \\ \textbf{implies } X1 = \emptyset, \end{gathered}$$

(64) $$[:X1,X2,X3,X4:] \text{ meets } [:Y1,Y2,Y3,Y4:]$$
$$\textbf{implies } X1 \text{ meets } Y1 \ \& \ X2 \text{ meets } Y2 \ \& \ X3 \text{ meets } Y3 \ \& \ X4 \text{ meets } Y4,$$

(65) $$[:\{x1\},\{x2\},\{x3\},\{x4\}:] = \{\langle x1,x2,x3,x4\rangle\},$$

(66) $[:X, Y:] \neq \emptyset$ **implies for** x **being** Element **of** $[:X, Y:]$ **holds** $x \neq x_{\mathbf{1}}$ & $x \neq x_{\mathbf{2}}$,

(67) $$x \in [:X, Y:] \textbf{ implies } x \neq x_{\mathbf{1}} \ \& \ x \neq x_{\mathbf{2}}.$$

For simplicity we adopt the following convention: $A1$ will denote an object of the type Subset **of** $X1$; $A2$ will denote an object of the type Subset **of** $X2$; $A3$ will denote an object of the type Subset **of** $X3$; $A4$ will denote an object of the type Subset **of** $X4$; x will denote an object of the type Element **of** $[:X1,X2,X3:]$. We now state a number of propositions:

(68) $$X1 \neq \emptyset \ \& \ X2 \neq \emptyset \ \& \ X3 \neq \emptyset \textbf{ implies}$$
$$\textbf{for } x1,x2,x3 \textbf{ st } x = \langle x1,x2,x3\rangle \textbf{ holds } x_{\mathbf{1}} = x1 \ \& \ x_{\mathbf{2}} = x2 \ \& \ x_{\mathbf{3}} = x3,$$

(69) $$X1 \neq \emptyset \ \&$$
$$X2 \neq \emptyset \ \& \ X3 \neq \emptyset \ \& \ (\textbf{for } xx1,xx2,xx3 \textbf{ st } x = \langle xx1,xx2,xx3\rangle \textbf{ holds } y1 = xx1)$$
$$\textbf{implies } y1 = x_{\mathbf{1}},$$

(70) $$X1 \neq \emptyset \ \&$$
$$X2 \neq \emptyset \ \& \ X3 \neq \emptyset \ \& \ (\textbf{for } xx1,xx2,xx3 \textbf{ st } x = \langle xx1,xx2,xx3\rangle \textbf{ holds } y2 = xx2)$$
$$\textbf{implies } y2 = x_{\mathbf{2}},$$

(71) $$X1 \neq \emptyset \ \&$$
$$X2 \neq \emptyset \ \& \ X3 \neq \emptyset \ \& \ (\textbf{for } xx1,xx2,xx3 \textbf{ st } x = \langle xx1,xx2,xx3\rangle \textbf{ holds } y3 = xx3)$$
$$\textbf{implies } y3 = x_{\mathbf{3}},$$

(72) $$z \in [:X1,X2,X3:]$$
$$\textbf{implies ex } x1,x2,x3 \textbf{ st } x1 \in X1 \ \& \ x2 \in X2 \ \& \ x3 \in X3 \ \& \ z = \langle x1,x2,x3\rangle,$$

(73) $$\langle x1,x2,x3\rangle \in [:X1,X2,X3:] \textbf{ iff } x1 \in X1 \ \& \ x2 \in X2 \ \& \ x3 \in X3,$$

(74) $$(\textbf{for } z \textbf{ holds}$$
$$z \in Z \textbf{ iff ex } x1,x2,x3 \textbf{ st } x1 \in X1 \ \& \ x2 \in X2 \ \& \ x3 \in X3 \ \& \ z = \langle x1,x2,x3\rangle)$$
$$\textbf{implies } Z = [:X1,X2,X3:],$$

(75) $$X1 \neq \emptyset \ \& \ X2 \neq \emptyset \ \& \ X3 \neq \emptyset \ \& \ Y1 \neq \emptyset \ \& \ Y2 \neq \emptyset \ \& \ Y3 \neq \emptyset \textbf{ implies}$$
$$\textbf{for } x \textbf{ being } \text{Element } \textbf{of } [:X1,X2,X3:], y \textbf{ being } \text{Element } \textbf{of } [:Y1,Y2,Y3:]$$
$$\textbf{holds } x = y \textbf{ implies } x_{\mathbf{1}} = y_{\mathbf{1}} \ \& \ x_{\mathbf{2}} = y_{\mathbf{2}} \ \& \ x_{\mathbf{3}} = y_{\mathbf{3}},$$

$$\textbf{for } x \textbf{ being } \text{Element } \textbf{of } [:X1,X2,X3:] \textbf{ st } x \in [:A1,A2,A3:] \textbf{ holds } x_{\mathbf{1}} \in A1 \;\&\; x_{\mathbf{2}} \in A2 \;\&\; x_{\mathbf{3}} \in A3, \tag{76}$$

$$X1 \subseteq Y1 \;\&\; X2 \subseteq Y2 \;\&\; X3 \subseteq Y3 \textbf{ implies } [:X1,X2,X3:] \subseteq [:Y1,Y2,Y3:]. \tag{77}$$

In the sequel x has the type Element **of** $[:X1,X2,X3,X4:]$. We now state a number of propositions:

$$X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset \textbf{ implies for } x1,x2,x3,x4 \textbf{ st } x = \langle x1,x2,x3,x4 \rangle \textbf{ holds } x_{\mathbf{1}} = x1 \;\&\; x_{\mathbf{2}} = x2 \;\&\; x_{\mathbf{3}} = x3 \;\&\; x_{\mathbf{4}} = x4, \tag{78}$$

$$X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset \;\&\; (\textbf{for } xx1,xx2,xx3,xx4 \textbf{ st } x = \langle xx1,xx2,xx3,xx4 \rangle \textbf{ holds } y1 = xx1) \textbf{ implies } y1 = x_{\mathbf{1}}, \tag{79}$$

$$X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset \;\&\; (\textbf{for } xx1,xx2,xx3,xx4 \textbf{ st } x = \langle xx1,xx2,xx3,xx4 \rangle \textbf{ holds } y2 = xx2) \textbf{ implies } y2 = x_{\mathbf{2}}, \tag{80}$$

$$X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset \;\&\; (\textbf{for } xx1,xx2,xx3,xx4 \textbf{ st } x = \langle xx1,xx2,xx3,xx4 \rangle \textbf{ holds } y3 = xx3) \textbf{ implies } y3 = x_{\mathbf{3}}, \tag{81}$$

$$X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset \;\&\; (\textbf{for } xx1,xx2,xx3,xx4 \textbf{ st } x = \langle xx1,xx2,xx3,xx4 \rangle \textbf{ holds } y4 = xx4) \textbf{ implies } y4 = x_{\mathbf{4}}, \tag{82}$$

$$z \in [:X1,X2,X3,X4:] \textbf{ implies ex } x1,x2,x3,x4 \textbf{ st } x1 \in X1 \;\&\; x2 \in X2 \;\&\; x3 \in X3 \;\&\; x4 \in X4 \;\&\; z = \langle x1,x2,x3,x4 \rangle, \tag{83}$$

$$\langle x1,x2,x3,x4 \rangle \in [:X1,X2,X3,X4:] \textbf{ iff } x1 \in X1 \;\&\; x2 \in X2 \;\&\; x3 \in X3 \;\&\; x4 \in X4, \tag{84}$$

$$(\textbf{for } z \textbf{ holds } z \in Z \textbf{ iff ex } x1,x2,x3,x4 \textbf{ st } x1 \in X1 \;\&\; x2 \in X2 \;\&\; x3 \in X3 \;\&\; x4 \in X4 \;\&\; z = \langle x1,x2,x3,x4 \rangle) \textbf{ implies } Z = [:X1,X2,X3,X4:], \tag{85}$$

$$X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset \;\&\; Y1 \neq \emptyset \;\&\; Y2 \neq \emptyset \;\&\; Y3 \neq \emptyset \;\&\; Y4 \neq \emptyset \textbf{ implies for } x \textbf{ being } \text{Element } \textbf{of } [:X1,X2,X3,X4:],\, y \textbf{ being } \text{Element } \textbf{of } [:Y1,Y2,Y3,Y4:] \textbf{ holds } x = y \textbf{ implies } x_{\mathbf{1}} = y_{\mathbf{1}} \;\&\; x_{\mathbf{2}} = y_{\mathbf{2}} \;\&\; x_{\mathbf{3}} = y_{\mathbf{3}} \;\&\; x_{\mathbf{4}} = y_{\mathbf{4}}, \tag{86}$$

(87) $$\textbf{for } x \textbf{ being } \mathrm{Element} \textbf{ of } [:X1,X2,X3,X4:] \textbf{ st } x \in [:A1,A2,A3,A4:] \textbf{ holds } x_{\mathbf{1}} \in A1 \ \& \ x_{\mathbf{2}} \in A2 \ \& \ x_{\mathbf{3}} \in A3 \ \& \ x_{\mathbf{4}} \in A4,$$

(88) $$X1 \subseteq Y1 \ \& \ X2 \subseteq Y2 \ \& \ X3 \subseteq Y3 \ \& \ X4 \subseteq Y4 \textbf{ implies } [:X1,X2,X3,X4:] \subseteq [:Y1,Y2,Y3,Y4:].$$

Let us consider $X1$, $X2$, $A1$, $A2$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[:A1,A2:] \qquad \text{is} \qquad \text{Subset } \textbf{of } [:X1,X2:].$$

Let us consider $X1$, $X2$, $X3$, $A1$, $A2$, $A3$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[:A1,A2,A3:] \qquad \text{is} \qquad \text{Subset } \textbf{of } [:X1,X2,X3:].$$

Let us consider $X1$, $X2$, $X3$, $X4$, $A1$, $A2$, $A3$, $A4$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[:A1,A2,A3,A4:] \qquad \text{is} \qquad \text{Subset } \textbf{of } [:X1,X2,X3,X4:].$$

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