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Basic Properties of Real Numbers

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Summary. Basic facts of arithmetics of real numbers are presented: definitions and properties of the complement element, the inverse element, subtraction and division; some basic properties of the set REAL (e.g. density), and the scheme of separation for sets of reals.

For simplicity we adopt the following convention: x, y, z, t will denote objects of the type Real; r will denote an object of the type Any. Let us consider x, y. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$x + y \quad \text{is} \quad \text{Real},$$

$$x \cdot y \quad \text{is} \quad \text{Real}.$$

One can prove the following propositions:

$$r \textbf{ is } \text{Real} \textbf{ iff } r \in \text{REAL}, \tag{1}$$

$$x + y = y + x, \tag{2}$$

$$x + (y + z) = (x + y) + z, \tag{3}$$

$$x + 0 = x \;\&\; 0 + x = x, \tag{4}$$

$$x \cdot y = y \cdot x, \tag{5}$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z, \tag{6}$$

$$x \cdot 1 = x \;\&\; 1 \cdot x = x, \tag{7}$$

$$(x + y) \cdot z = x \cdot z + y \cdot z \;\&\; z \cdot (x + y) = z \cdot x + z \cdot y, \tag{8}$$

(9) $z \neq 0 \;\&\; x \neq y$ **implies** $x \cdot z \neq y \cdot z \;\&\; z \cdot x \neq y \cdot z \;\&\; z \cdot x \neq z \cdot y \;\&\; x \cdot z \neq z \cdot y$,

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(10) $$z + x = z + y \textbf{ or } x + z = y + z \textbf{ or } z + x = y + z \textbf{ or } x + z = z + y \textbf{ implies } x = y,$$

(11) $$x \neq y \textbf{ iff } x + z \neq y + z,$$

(12) $$z \neq 0 \;\&\; (x \cdot z = y \cdot z \textbf{ or } z \cdot x = z \cdot y \textbf{ or } x \cdot z = z \cdot y \textbf{ or } z \cdot x = y \cdot z) \textbf{ implies } x = y.$$

We now define two new functors. Let us consider x. The functor

$$-x,$$

with values of the type Real, is defined by

$$x + \textbf{it} = 0.$$

Assume that the following holds

$$x \neq 0.$$

The functor

$$x^{-1},$$

yields the type Real and is defined by

$$x \cdot \textbf{it} = 1.$$

We now define two new functors. Let us consider x, y. The functor

$$x - y,$$

yields the type Real and is defined by

$$\textbf{it} = x + (-y).$$

Assume that the following holds

$$y \neq 0.$$

The functor

$$x/y,$$

yields the type Real and is defined by

$$\textbf{it} = x \cdot y^{-1}.$$

The following propositions are true:

(13) $$x + -x = 0 \;\&\; -x + x = 0,$$

(14) $$x - y = x + -y,$$

(15) $$x \neq 0 \textbf{ implies } x \cdot x^{-1} = 1 \ \& \ x^{-1} \cdot x = 1,$$

(16) $$y \neq 0 \textbf{ implies } x/y = x \cdot y^{-1} \ \& \ x/y = y^{-1} \cdot x,$$

(17) $$x + y - z = x + (y - z),$$

(18) $$-(-x) = x,$$

(19) $$0 - x = -x,$$

(20) $$x \cdot 0 = 0 \ \& \ 0 \cdot x = 0,$$

(21) $$(-x) \cdot y = -(x \cdot y) \ \& \ x \cdot (-y) = -(x \cdot y) \ \& \ (-x) \cdot y = x \cdot (-y),$$

(22) $$x \neq 0 \textbf{ iff } -x \neq 0,$$

(23) $$x \cdot y = 0 \textbf{ iff } x = 0 \textbf{ or } y = 0,$$

(24) $$x \neq 0 \ \& \ y \neq 0 \textbf{ implies } x^{-1} \cdot y^{-1} = (x \cdot y)^{-1},$$

(25) $$x - 0 = x,$$

(26) $$-0 = 0,$$

(27) $$x - (y + z) = x - y - z,$$

(28) $$x - (y - z) = x - y + z,$$

(29) $$x \cdot (y - z) = x \cdot y - x \cdot z \ \& \ (y - z) \cdot x = y \cdot x - z \cdot x,$$

(30) $$x + z = y \textbf{ implies } x = y - z \ \& \ z = y - x,$$

(31) $$x \neq 0 \textbf{ implies } x^{-1} \neq 0,$$

(32) $$x \neq 0 \textbf{ implies } x^{-1\,-1} = x,$$

(33) $$x \neq 0 \textbf{ implies } 1/x = x^{-1} \ \& \ 1/x^{-1} = x,$$

(34) $$x \neq 0 \textbf{ implies } x \cdot (1/x) = 1 \ \& \ (1/x) \cdot x = 1,$$

(35) $$y \neq 0 \ \& \ t \neq 0 \textbf{ implies } (x/y) \cdot (z/t) = (x \cdot z)/(y \cdot t),$$

(36) $$x - x = 0,$$

(37) $$x \neq 0 \textbf{ implies } x/x = 1,$$

(38) $$y \neq 0 \ \& \ z \neq 0 \textbf{ implies } x/y = (x \cdot z)/(y \cdot z),$$

(39) $$y \neq 0 \textbf{ implies } -x/y = (-x)/y \ \& \ x/(-y) = -x/y,$$

(40) $$z \neq 0 \textbf{ implies } x/z + y/z = (x+y)/z \mathrel{\&} x/z - y/z = (x-y)/z,$$

(41) $$y \neq 0 \mathrel{\&} t \neq 0$$
$$\textbf{implies } x/y + z/t = (x \cdot t + z \cdot y)/(y \cdot t) \mathrel{\&} x/y - z/t = (x \cdot t - z \cdot y)/(y \cdot t),$$

(42) $$y \neq 0 \mathrel{\&} z \neq 0 \textbf{ implies } x/(y/z) = (x \cdot z)/y,$$

(43) $$y \neq 0 \textbf{ implies } x/y \cdot y = x,$$

(44) $$\textbf{for } x,y \textbf{ ex } z \textbf{ st } x = y + z \mathrel{\&} x = z + y,$$

(45) $$\textbf{for } x,y \textbf{ st } y \neq 0 \textbf{ ex } z \textbf{ st } x = y \cdot z \mathrel{\&} x = z \cdot y,$$

(46) $$x \leq y \mathrel{\&} y \leq x \textbf{ implies } x = y,$$

(47) $$x \leq y \mathrel{\&} y \leq z \textbf{ implies } x \leq z,$$

(48) $$x \leq y \textbf{ or } y \leq x,$$

(49) $$x \leq y \textbf{ implies } x + z \leq y + z \mathrel{\&} x - z \leq y - z,$$

(50) $$x \leq y \textbf{ iff } -y \leq -x,$$

(51) $$x \leq y \mathrel{\&} 0 \leq z \textbf{ implies } x \cdot z \leq y \cdot z \mathrel{\&} z \cdot x \leq z \cdot y \mathrel{\&} z \cdot x \leq y \cdot z \mathrel{\&} x \cdot z \leq z \cdot y,$$

(52) $$x \leq y \mathrel{\&} z \leq 0 \textbf{ implies } y \cdot z \leq x \cdot z \mathrel{\&} z \cdot y \leq z \cdot x \mathrel{\&} y \cdot z \leq z \cdot x \mathrel{\&} z \cdot y \leq x \cdot z,$$

(53) $$x \leq y \textbf{ iff } x + z \leq y + z,$$

(54) $$x \leq y \textbf{ iff } x - z \leq y - z,$$

(55) $$x \leq y \mathrel{\&} z \leq t$$
$$\textbf{implies } x + z \leq y + t \mathrel{\&} x + z \leq t + y \mathrel{\&} z + x \leq t + y \mathrel{\&} z + x \leq y + t,$$

(56) $$x \leq x.$$

Let us consider x, y. The predicate

$$x < y \qquad \text{is defined by} \qquad x \leq y \mathrel{\&} x \neq y.$$

One can prove the following propositions:

(57) $$x < y \textbf{ iff } x \leq y \mathrel{\&} x \neq y,$$

(58) $$x \leq y \mathrel{\&} y < z \textbf{ or } x < y \mathrel{\&} y \leq z \textbf{ or } x < y \mathrel{\&} y < z \textbf{ implies } x < z,$$

(59) $$x < y \textbf{ implies } x + z < y + z$$
$$\mathrel{\&} x - z < y - z \mathrel{\&} z + x < z + y \mathrel{\&} x + z < z + y \mathrel{\&} z + x < y + z,$$

(60) $$x + z < y + z \ \textbf{or}\ z + x < z + y \ \textbf{or}\ x + z < z + y \ \textbf{or}\ z + x < y + z \ \textbf{or}\ x - z < y - z \ \textbf{implies}\ x < y,$$

(61) $$x \neq y \ \textbf{implies}\ x < y \ \textbf{or}\ y < x,$$

(62) $$\textbf{not}\ x < y \ \textbf{iff}\ y \leq x,$$

(63) $$x < y \ \textbf{or}\ y < x \ \textbf{or}\ x = y,$$

(64) $$x < y \ \textbf{implies not}\ y < x,$$

(65) $$0 < 1,$$

(66) $$x < 0 \ \textbf{iff}\ 0 < -x,$$

(67) $$x < y \ \&\ z \leq t \ \textbf{or}\ x \leq y \ \&\ z < t \ \textbf{or}\ x < y \ \&\ z < t \ \textbf{implies}\ x + z < y + t \ \&\ z + x < y + t \ \&\ z + x < t + y \ \&\ x + z < t + y,$$

(68) $$x < y \ \textbf{iff}\ -y < -x,$$

(69) $$\textbf{for}\ x,y \ \textbf{st}\ 0 < x \ \textbf{holds}\ y < y + x,$$

(70) $$0 < z \ \&\ x < y \ \textbf{implies}\ x \cdot z < y \cdot z \ \&\ z \cdot x < z \cdot y \ \&\ x \cdot z < z \cdot y \ \&\ z \cdot x < y \cdot z,$$

(71) $$z < 0 \ \&\ x < y \ \textbf{implies}\ y \cdot z < x \cdot z \ \&\ z \cdot y < z \cdot x \ \&\ y \cdot z < z \cdot x \ \&\ z \cdot y < x \cdot z,$$

(72) $$0 < z \ \textbf{implies}\ 0 < z^{-1},$$

(73) $$0 < z \ \textbf{implies}\ (x < y \ \textbf{iff}\ x/z < y/z),$$

(74) $$z < 0 \ \textbf{implies}\ (x < y \ \textbf{iff}\ y/z < x/z),$$

(75) $$x < y \ \textbf{implies ex}\ z \ \textbf{st}\ x < z \ \&\ z < y,$$

(76) $$\textbf{for}\ x \ \textbf{ex}\ y \ \textbf{st}\ x < y,$$

(77) $$\textbf{for}\ x \ \textbf{ex}\ y \ \textbf{st}\ y < x,$$

(78) $$\textbf{for}\ X,Y \ \textbf{being}\ \text{Subset}\ \textbf{of}\ \text{REAL}\ \textbf{st}\ (\textbf{ex}\ x \ \textbf{st}\ x \in X) \ \&\ (\textbf{ex}\ x \ \textbf{st}\ x \in Y) \ \&\ \textbf{for}\ x,y \ \textbf{st}\ x \in X \ \&\ y \in Y \ \textbf{holds}\ x \leq y \ \textbf{ex}\ z \ \textbf{st for}\ x,y \ \textbf{st}\ x \in X \ \&\ y \in Y \ \textbf{holds}\ x \leq z \ \&\ z \leq y.$$

The scheme *SepReal* concerns a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{ex}\ X \ \textbf{being set of}\ \text{Real}\ \textbf{st for}\ x \ \textbf{holds}\ x \in X \ \textbf{iff}\ \mathcal{P}[x]$$

for all values of the parameter.

The following propositions are true:

(79) $$y = -x \textbf{ iff } x + y = 0,$$

(80) $$\textbf{for } x, y \textbf{ st } x \neq 0 \textbf{ holds } y = x^{-1} \textbf{ iff } x \cdot y = 1,$$

(81) $$\textbf{for } x, y \textbf{ st } x \neq 0 \ \& \ y \neq 0 \textbf{ holds } (x/y)^{-1} = y/x,$$

(82) $$\textbf{for } x, y, z, t \textbf{ st } y \neq 0 \ \& \ z \neq 0 \ \& \ t \neq 0 \textbf{ holds } (x/y)/(z/t) = (x \cdot t)/(y \cdot z),$$

(83) $$-(x - y) = y - x,$$

(84) $$x + y \leq z \textbf{ iff } x \leq z - y,$$

(85) $$x + y \leq z \textbf{ iff } y \leq z - x,$$

(86) $$x \leq y + z \textbf{ iff } x - y \leq z,$$

(87) $$x \leq y + z \textbf{ iff } x - z \leq y,$$

(88) $$x + y < z \textbf{ iff } x < z - y,$$

(89) $$x + y < z \textbf{ iff } y < z - x,$$

(90) $$x < z + y \textbf{ iff } x - z < y,$$

(91) $$x < y + z \textbf{ iff } x - z < y,$$

(92) $$(x \leq y \ \& \ z \leq t \textbf{ implies } x - t \leq y - z)$$
$$\& \ (x < y \ \& \ z \leq t \textbf{ or } x \leq y \ \& \ z < t \textbf{ or } x < y \ \& \ z < t \textbf{ implies } x - t < y - z),$$

(93) $$0 \leq x \cdot x.$$

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