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Relations Defined on Sets

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Summary. The article includes theorems concerning properties of relations defined as a subset of the Cartesian product of two sets (mode Relation of X,Y where X,Y are sets). Some notions, introduced in [3] such as domain, codomain, field of a relation, composition of relations, image and inverse image of a set under a relation are redefined.

The articles [1], [2], and [3] provide the terminology and notation for this paper. For simplicity we adopt the following convention: A, B, X, $X1$, Y, $Y1$, Z will denote objects of the type set; a, x, y will denote objects of the type Any. Let us consider X, Y. The mode

$$\text{Relation } \textbf{of } X, Y,$$

which widens to the type Relation, is defined by

$$\textbf{it} \subseteq [:X, Y:].$$

The following proposition is true

(1) **for** R **being** Relation **holds** $R \subseteq [:X, Y:]$ **iff** R **is** Relation **of** X, Y.

In the sequel P, R will denote objects of the type Relation **of** X, Y. The following propositions are true:

(2) $A \subseteq R$ **implies** $A \subseteq [:X, Y:]$,

(3) $A \subseteq [:X, Y:]$ **implies** A **is** Relation **of** X, Y,

(4) $A \subseteq R$ **implies** A **is** Relation **of** X, Y,

(5) $[:X, Y:]$ **is** Relation **of** X, Y,

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(6) $$a \in R \textbf{ implies } \textbf{ex } x{,}y \textbf{ st } a = \langle x, y\rangle \ \&\ x \in X \ \&\ y \in Y,$$

(7) $$\langle x, y\rangle \in R \textbf{ implies } x \in X \ \&\ y \in Y,$$

(8) $$x \in X \ \&\ y \in Y \textbf{ implies } \{\langle x, y\rangle\} \textbf{ is } \mathrm{Relation} \textbf{ of } X, Y,$$

(9) $$\textbf{for } R \textbf{ being } \mathrm{Relation} \textbf{ st } \mathrm{dom}\, R \subseteq X \textbf{ holds } R \textbf{ is } \mathrm{Relation} \textbf{ of } X, \mathrm{rng}\, R,$$

(10) $$\textbf{for } R \textbf{ being } \mathrm{Relation} \textbf{ st } \mathrm{rng}\, R \subseteq Y \textbf{ holds } R \textbf{ is } \mathrm{Relation} \textbf{ of } \mathrm{dom}\, R{,}Y,$$

(11) $$\textbf{for } R \textbf{ being } \mathrm{Relation}$$
$$\textbf{st } \mathrm{dom}\, R \subseteq X \ \&\ \mathrm{rng}\, R \subseteq Y \textbf{ holds } R \textbf{ is } \mathrm{Relation} \textbf{ of } X, Y,$$

(12) $$\mathrm{dom}\, R \subseteq X \ \&\ \mathrm{rng}\, R \subseteq Y,$$

(13) $$\mathrm{dom}\, R \subseteq X1 \textbf{ implies } R \textbf{ is } \mathrm{Relation} \textbf{ of } X1{,}Y,$$

(14) $$\mathrm{rng}\, R \subseteq Y1 \textbf{ implies } R \textbf{ is } \mathrm{Relation} \textbf{ of } X, Y1,$$

(15) $$X \subseteq X1 \textbf{ implies } R \textbf{ is } \mathrm{Relation} \textbf{ of } X1{,}Y,$$

(16) $$Y \subseteq Y1 \textbf{ implies } R \textbf{ is } \mathrm{Relation} \textbf{ of } X, Y1,$$

(17) $$X \subseteq X1 \ \&\ Y \subseteq Y1 \textbf{ implies } R \textbf{ is } \mathrm{Relation} \textbf{ of } X1{,}Y1.$$

Let us consider X, Y, P, R. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$P \cup R \quad \text{is} \quad \mathrm{Relation} \textbf{ of } X, Y,$$

$$P \cap R \quad \text{is} \quad \mathrm{Relation} \textbf{ of } X, Y,$$

$$P \setminus R \quad \text{is} \quad \mathrm{Relation} \textbf{ of } X, Y.$$

We now state a proposition

(18) $$R \cap [:X, Y:] = R.$$

Let us consider X, Y, R. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$\mathrm{dom}\, R \quad \text{is} \quad \mathrm{Subset} \textbf{ of } X,$$

$$\mathrm{rng}\, R \quad \text{is} \quad \mathrm{Subset} \textbf{ of } Y.$$

Next we state several propositions:

(19) $$\mathrm{field}\, R \subseteq X \cup Y,$$

(20) $$\textbf{for } R \textbf{ being } \mathrm{Relation} \textbf{ holds } R \textbf{ is } \mathrm{Relation} \textbf{ of } \mathrm{dom}\, R{,}\mathrm{rng}\, R,$$

(21) $\operatorname{dom} R \subseteq X1$ & $\operatorname{rng} R \subseteq Y1$ **implies** R **is** Relation **of** $X1$,$Y1$,

(22) (**for** x **st** $x \in X$ **ex** y **st** $\langle x, y\rangle \in R$) **iff** $\operatorname{dom} R = X$,

(23) (**for** y **st** $y \in Y$ **ex** x **st** $\langle x, y\rangle \in R$) **iff** $\operatorname{rng} R = Y$.

Let us consider X, Y, R. Let us note that it makes sense to consider the following functor on a restricted area. Then

$R\tilde{}$ is Relation **of** Y, X.

The arguments of the notions defined below are the following: X, Y, Z which are objects of the type reserved above; P which is an object of the type Relation **of** X, Y; R which is an object of the type Relation **of** Y, Z. Let us note that it makes sense to consider the following functor on a restricted area. Then

$P \cdot R$ is Relation **of** X, Z.

One can prove the following propositions:

(24) $\operatorname{dom}(R\tilde{}) = \operatorname{rng} R$ & $\operatorname{rng}(R\tilde{}) = \operatorname{dom} R$,

(25) $\varnothing$ **is** Relation **of** X, Y,

(26) R **is** Relation **of** $\emptyset$,Y **implies** $R = \varnothing$,

(27) R **is** Relation **of** $X, \emptyset$ **implies** $R = \varnothing$,

(28) $\triangle X \subseteq [: X, X :]$,

(29) $\triangle X$ **is** Relation **of** X, X,

(30) $\triangle A \subseteq R$ **implies** $A \subseteq \operatorname{dom} R$ & $A \subseteq \operatorname{rng} R$,

(31) $\triangle X \subseteq R$ **implies** $X = \operatorname{dom} R$ & $X \subseteq \operatorname{rng} R$,

(32) $\triangle Y \subseteq R$ **implies** $Y \subseteq \operatorname{dom} R$ & $Y = \operatorname{rng} R$.

Let us consider X, Y, R, A. Let us note that it makes sense to consider the following functor on a restricted area. Then

$R \mid A$ is Relation **of** X, Y.

Let us consider X, Y, B, R. Let us note that it makes sense to consider the following functor on a restricted area. Then

$B \mid R$ is Relation **of** X, Y.

The following four propositions are true:

(33) $R \mid X1$ **is** Relation **of** $X1$,Y,

(34) $$X \subseteq X1 \textbf{ implies } R \mid X1 = R,$$

(35) $$Y1 \mid R \textbf{ is } \text{Relation } \textbf{of } X, Y1,$$

(36) $$Y \subseteq Y1 \textbf{ implies } Y1 \mid R = R.$$

Let us consider X, Y, R, A. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$R^{\circ} A \quad \text{is} \quad \text{Subset } \textbf{of } Y,$$

$$R^{-1} A \quad \text{is} \quad \text{Subset } \textbf{of } X.$$

Next we state three propositions:

(37) $$R^{\circ} A \subseteq Y \ \& \ R^{-1} A \subseteq X,$$

(38) $$R^{\circ} X = \operatorname{rng} R \ \& \ R^{-1} Y = \operatorname{dom} R,$$

(39) $$R^{\circ} (R^{-1} Y) = \operatorname{rng} R \ \& \ R^{-1} (R^{\circ} X) = \operatorname{dom} R.$$

The scheme *Rel_On_Set_Ex* deals with a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } R \textbf{ being } \text{Relation } \textbf{of } \mathcal{A},\mathcal{B} \textbf{ st for } x,y \textbf{ holds } \langle x, y\rangle \in R \textbf{ iff } x \in \mathcal{A} \ \& \ y \in \mathcal{B} \ \& \ \mathcal{P}[x, y]$$

for all values of the parameters.

Let us consider X.

$$\text{Relation } \textbf{of } X \quad \text{stands for} \quad \text{Relation } \textbf{of } X, X.$$

We now state three propositions:

(40) $$\textbf{for } R \textbf{ being } \text{Relation } \textbf{of } X, X \textbf{ holds } R \subseteq [:X, X:] \textbf{ iff } R \textbf{ is } \text{Relation } \textbf{of } X,$$

(41) $$[:X, X:] \textbf{ is } \text{Relation } \textbf{of } X,$$

(42) $$\textbf{for } R \textbf{ being } \text{Relation } \textbf{of } X, X \textbf{ holds } R \textbf{ is } \text{Relation } \textbf{of } X.$$

In the sequel R denotes an object of the type Relation **of** X. One can prove the following propositions:

(43) $$\triangle X \textbf{ is } \text{Relation } \textbf{of } X,$$

(44) $$\triangle X \subseteq R \textbf{ implies } X = \operatorname{dom} R \ \& \ X = \operatorname{rng} R,$$

(45) $$R \cdot (\triangle X) = R \ \& \ (\triangle X) \cdot R = R.$$

For simplicity we adopt the following convention: D, $D1$, $D2$, E, F denote objects of the type DOMAIN; R denotes an object of the type Relation **of** D, E; x denotes an object of the type Element **of** D; y denotes an object of the type Element **of** E. We now state a proposition

(46) $$\triangle D \neq \emptyset\,.$$

Let us consider D, E, R. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$\operatorname{dom} R \quad \text{is} \quad \text{Element } \textbf{of} \operatorname{bool} D,$$

$$\operatorname{rng} R \quad \text{is} \quad \text{Element } \textbf{of} \operatorname{bool} E.$$

Next we state several propositions:

(47) $$\textbf{for}\; x \;\textbf{being}\; \text{Element } \textbf{of}\; D$$
$$\textbf{holds}\; x \in \operatorname{dom} R \;\textbf{iff}\; \textbf{ex}\; y \;\textbf{being}\; \text{Element } \textbf{of}\; E \;\textbf{st}\; \langle x, y\rangle \in R,$$

(48) $$\textbf{for}\; y \;\textbf{being}\; \text{Element } \textbf{of}\; E$$
$$\textbf{holds}\; y \in \operatorname{rng} R \;\textbf{iff}\; \textbf{ex}\; x \;\textbf{being}\; \text{Element } \textbf{of}\; D \;\textbf{st}\; \langle x, y\rangle \in R,$$

(49) $$\textbf{for}\; x \;\textbf{being}\; \text{Element } \textbf{of}\; D$$
$$\textbf{holds}\; x \in \operatorname{dom} R \;\textbf{implies}\; \textbf{ex}\; y \;\textbf{being}\; \text{Element } \textbf{of}\; E \;\textbf{st}\; y \in \operatorname{rng} R,$$

(50) $$\textbf{for}\; y \;\textbf{being}\; \text{Element } \textbf{of}\; E$$
$$\textbf{holds}\; y \in \operatorname{rng} R \;\textbf{implies}\; \textbf{ex}\; x \;\textbf{being}\; \text{Element } \textbf{of}\; D \;\textbf{st}\; x \in \operatorname{dom} R,$$

(51) $$\textbf{for}\; P \;\textbf{being}\; \text{Relation } \textbf{of}\; D, E, R \;\textbf{being}\; \text{Relation } \textbf{of}\; E, F$$
$$\textbf{for}\; x \;\textbf{being}\; \text{Element } \textbf{of}\; D, z \;\textbf{being}\; \text{Element } \textbf{of}\; F$$
$$\textbf{holds}\; \langle x, z\rangle \in P \cdot R \;\textbf{iff}\; \textbf{ex}\; y \;\textbf{being}\; \text{Element } \textbf{of}\; E \;\textbf{st}\; \langle x, y\rangle \in P \;\&\; \langle y, z\rangle \in R.$$

Let us consider D, E, R, $D1$. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$R^{\circ}\, D1 \quad \text{is} \quad \text{Element } \textbf{of} \operatorname{bool} E,$$

$$R^{-1}\, D1 \quad \text{is} \quad \text{Element } \textbf{of} \operatorname{bool} D.$$

We now state two propositions:

(52) $$y \in R^{\circ}\, D1 \;\textbf{iff}\; \textbf{ex}\; x \;\textbf{being}\; \text{Element } \textbf{of}\; D \;\textbf{st}\; \langle x, y\rangle \in R \;\&\; x \in D1,$$

(53) $$x \in R^{-1}\, D2 \;\textbf{iff}\; \textbf{ex}\; y \;\textbf{being}\; \text{Element } \textbf{of}\; E \;\textbf{st}\; \langle x, y\rangle \in R \;\&\; y \in D2.$$

The scheme *Rel_On_Dom_Ex* concerns a constant $\mathcal{A}$ that has the type DOMAIN, a constant $\mathcal{B}$ that has the type DOMAIN and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex}\; R \;\textbf{being}\; \text{Relation } \textbf{of}\; \mathcal{A}, \mathcal{B} \;\textbf{st}\; \textbf{for}\; x \;\textbf{being}\; \text{Element } \textbf{of}\; \mathcal{A}, y \;\textbf{being}\; \text{Element } \textbf{of}\; \mathcal{B}$$
$$\textbf{holds}\; \langle x, y\rangle \in R \;\textbf{iff}\; x \in \mathcal{A} \;\&\; y \in \mathcal{B} \;\&\; \mathcal{P}[x, y]$$

for all values of the parameters.

References

[1] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1, 1990.

[2] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1, 1990.

[3] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1, 1990.

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