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Tarski Grothendieck Set Theory

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Summary. This is the first part of the axiomatics of the Mizar system. It includes the axioms of the Tarski Grothendieck set theory. They are: the axiom stating that everything is a set, the extensionality axiom, the definitional axiom of the singleton, the definitional axiom of the pair, the definitional axiom of the union of a family of sets, the definitional axiom of the boolean (the power set) of a set, the regularity axiom, the definitional axiom of the ordered pair, the Tarski's axiom A introduced in [2] (see also [1]), and the Frænkel scheme. Also, the definition of equinumerosity is introduced.

For simplicity we adopt the following convention: x, y, z, u will denote objects of the type Any; N, M, X, Y, Z will denote objects of the type set. Next we state two axioms:

(1) $$x \ \textbf{is} \ \text{set},$$

(2) $$(\textbf{for}\ x\ \textbf{holds}\ x \in X\ \textbf{iff}\ x \in Y)\ \textbf{implies}\ X = Y.$$

We now introduce two functors. Let us consider y. The functor

$$\{y\},$$

with values of the type set, is defined by

$$x \in \textbf{it}\ \textbf{iff}\ x = y.$$

Let us consider z. The functor

$$\{y, z\},$$

with values of the type set, is defined by

$$x \in \textbf{it}\ \textbf{iff}\ x = y\ \textbf{or}\ x = z.$$

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The following axioms hold:

$$X = \{y\} \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff } x = y, \tag{3}$$

$$X = \{y, z\} \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff } x = y \textbf{ or } x = z. \tag{4}$$

Let us consider X, Y. The predicate

$$X \subseteq Y \quad \text{is defined by} \quad x \in X \textbf{ implies } x \in Y.$$

Let us consider X. The functor

$$\bigcup X,$$

with values of the type set, is defined by

$$x \in \textbf{it iff ex } Y \textbf{ st } x \in Y \ \& \ Y \in X.$$

Then we get

$$X = \bigcup Y \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff ex } Z \textbf{ st } x \in Z \ \& \ Z \in Y, \tag{5}$$

$$X = \operatorname{bool} Y \textbf{ iff for } Z \textbf{ holds } Z \in X \textbf{ iff } Z \subseteq Y, \tag{6}$$

The regularity axiom claims that

$$x \in X \textbf{ implies ex } Y \textbf{ st } Y \in X \ \& \textbf{ not ex } x \textbf{ st } x \in X \ \& \ x \in Y. \tag{7}$$

The scheme *Fraenkel* deals with a constant $\mathcal{A}$ that has the type set and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } X \textbf{ st for } x \textbf{ holds } x \in X \textbf{ iff ex } y \textbf{ st } y \in \mathcal{A} \ \& \ \mathcal{P}[y, x]$$

provided the parameters satisfy the following condition:

- $$\textbf{for } x,y,z \textbf{ st } \mathcal{P}[x, y] \ \& \ \mathcal{P}[x, z] \textbf{ holds } y = z.$$

Let us consider x, y. The functor

$$\langle x, y\rangle,$$

is defined by

$$\textbf{it} = \{\{x, y\},\{x\}\}.$$

According to the definition

$$\langle x, y\rangle = \{\{x, y\},\{x\}\}. \tag{8}$$

Let us consider X, Y. The predicate

$$X \approx Y$$

is defined by

$$\textbf{ex}\, Z \textbf{ st } (\textbf{for}\, x \textbf{ st } x \in X \textbf{ ex}\, y \textbf{ st } y \in Y \;\&\; \langle x, y \rangle \in Z) \;\&$$
$$(\textbf{for}\, y \textbf{ st } y \in Y \textbf{ ex}\, x \textbf{ st } x \in X \;\&\; \langle x, y \rangle \in Z)$$
$$\&\; \textbf{for}\, x, y, z, u \textbf{ st } \langle x, y \rangle \in Z \;\&\; \langle z, u \rangle \in Z \textbf{ holds } x = z \textbf{ iff } y = u.$$

The Tarski's axiom A claims that

(9) $$\textbf{ex}\, M \textbf{ st } N \in M \;\&\; (\textbf{for}\, X, Y \textbf{ holds } X \in M \;\&\; Y \subseteq X \textbf{ implies } Y \in M) \;\&$$
$$(\textbf{for}\, X \textbf{ holds } X \in M \textbf{ implies } \mathrm{bool}\, X \in M)$$
$$\&\; \textbf{for}\, X \textbf{ holds } X \subseteq M \textbf{ implies } X \approx M \textbf{ or } X \in M.$$

References

[1] Alfred Tarski. On well-ordered subsets of any set. *Fundamenta Mathematicae*, 32:176–183, 1939.

[2] Alfred Tarski. Über Unerreaichbare Kardinalzahlen. *Fundamenta Mathematicae*, 30:176–183, 1938.

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