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Some Basic Properties of Sets

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Summary. In this article some basic theorems about singletons, pairs, power sets, unions of families of sets, and the cartesian product of two sets are proved.

The articles [1] and [2] provide the terminology and notation for this paper. One can prove the following propositions:

(1) $$\mathrm{bool}\,\emptyset = \{\emptyset\},$$

(2) $$\bigcup \emptyset = \emptyset.$$

For simplicity we adopt the following convention: x, $x1$, $x2$, y, $y1$, $y2$, z will denote objects of the type Any; A, B, X, $X1$, $X2$, Y, $Y1$, $Y2$, Z will denote objects of the type set. One can prove the following propositions:

(3) $$\{x\} \neq \emptyset,$$

(4) $$\{x, y\} \neq \emptyset,$$

(5) $$\{x\} = \{x, x\},$$

(6) $$\{x\} = \{y\} \textbf{ implies } x = y,$$

(7) $$\{x1,x2\} = \{x2,x1\},$$

(8) $$\{x\} = \{y1,y2\} \textbf{ implies } x = y1 \ \& \ x = y2,$$

(9) $$\{x\} = \{y1,y2\} \textbf{ implies } y1 = y2,$$

(10) $$\{x1,x2\} = \{y1,y2\} \textbf{ implies } (x1 = y1 \textbf{ or } x1 = y2) \ \& \ (x2 = y1 \textbf{ or } x2 = y2),$$

(11) $$\{x1,x2\} = \{x1\} \cup \{x2\},$$

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$$\{x\} \subseteq \{x, y\} \ \& \ \{y\} \subseteq \{x, y\}, \tag{12}$$

$$\{x\} \cup \{y\} = \{x\} \textbf{ or } \{x\} \cup \{y\} = \{y\} \textbf{ implies } x = y, \tag{13}$$

$$\{x\} \cup \{x, y\} = \{x, y\} \ \& \ \{x, y\} \cup \{x\} = \{x, y\}, \tag{14}$$

$$\{y\} \cup \{x, y\} = \{x, y\} \ \& \ \{x, y\} \cup \{y\} = \{x, y\}, \tag{15}$$

$$\{x\} \cap \{y\} = \emptyset \textbf{ or } \{y\} \cap \{x\} = \emptyset \textbf{ implies } x \neq y, \tag{16}$$

$$x \neq y \textbf{ implies } \{x\} \cap \{y\} = \emptyset \ \& \ \{y\} \cap \{x\} = \emptyset, \tag{17}$$

$$\{x\} \cap \{y\} = \{x\} \textbf{ or } \{x\} \cap \{y\} = \{y\} \textbf{ implies } x = y, \tag{18}$$

$$\begin{gathered} \{x\} \cap \{x, y\} = \{x\} \\ \& \ \{y\} \cap \{x, y\} = \{y\} \ \& \ \{x, y\} \cap \{x\} = \{x\} \ \& \ \{x, y\} \cap \{y\} = \{y\}, \end{gathered} \tag{19}$$

$$\{x\} \setminus \{y\} = \{x\} \textbf{ iff } x \neq y, \tag{20}$$

$$\{x\} \setminus \{y\} = \emptyset \textbf{ implies } x = y, \tag{21}$$

$$\{x\} \setminus \{x, y\} = \emptyset \ \& \ \{y\} \setminus \{x, y\} = \emptyset, \tag{22}$$

$$x \neq y \textbf{ implies } \{x, y\} \setminus \{y\} = \{x\} \ \& \ \{x, y\} \setminus \{x\} = \{y\}, \tag{23}$$

$$\{x\} \subseteq \{y\} \textbf{ implies } \{x\} = \{y\}, \tag{24}$$

$$\{z\} \subseteq \{x, y\} \textbf{ implies } z = x \textbf{ or } z = y, \tag{25}$$

$$\{x, y\} \subseteq \{z\} \textbf{ implies } x = z \ \& \ y = z, \tag{26}$$

$$\{x, y\} \subseteq \{z\} \textbf{ implies } \{x, y\} = \{z\}, \tag{27}$$

$$\{x1, x2\} \subseteq \{y1, y2\} \textbf{ implies } (x1 = y1 \textbf{ or } x1 = y2) \ \& \ (x2 = y1 \textbf{ or } x2 = y2), \tag{28}$$

$$x \neq y \textbf{ implies } \{x\} \dot{-} \{y\} = \{x, y\}, \tag{29}$$

$$\mathrm{bool}\{x\} = \{\emptyset, \{x\}\}, \tag{30}$$

$$\bigcup \{x\} = x, \tag{31}$$

$$\bigcup \{\{x\}, \{y\}\} = \{x, y\}, \tag{32}$$

$$\langle x1, x2 \rangle = \langle y1, y2 \rangle \textbf{ implies } x1 = y1 \ \& \ x2 = y2, \tag{33}$$

$$\langle x, y \rangle \in [\!:\{x1\}, \{y1\}:\!] \textbf{ iff } x = x1 \ \& \ y = y1, \tag{34}$$

$$[\!:\{x\}, \{y\}:\!] = \{\langle x, y \rangle\}, \tag{35}$$

(36) $$[:\{x\},\{y,z\}:] = \{\langle x,y\rangle,\langle x,z\rangle\} \ \&\ [:\{x,y\},\{z\}:] = \{\langle x,z\rangle,\langle y,z\rangle\},$$

(37) $$\{x\} \subseteq X \textbf{ iff } x \in X,$$

(38) $$\{x1,x2\} \subseteq Z \textbf{ iff } x1 \in Z \ \&\ x2 \in Z,$$

(39) $$Y \subseteq \{x\} \textbf{ iff } Y = \emptyset \textbf{ or } Y = \{x\},$$

(40) $$Y \subseteq X \ \&\ \textbf{not}\, x \in Y \textbf{ implies } Y \subseteq X \setminus \{x\},$$

(41) $$X \neq \{x\} \ \&\ x \in X \textbf{ implies ex}\, y \textbf{ st } y \in X \ \&\ y \neq x,$$

(42) $$Z \subseteq \{x1,x2\} \textbf{ iff } Z = \emptyset \textbf{ or } Z = \{x1\} \textbf{ or } Z = \{x2\} \textbf{ or } Z = \{x1,x2\},$$

(43) $$\{z\} = X \cup Y$$
$$\textbf{implies } X = \{z\} \ \&\ Y = \{z\} \textbf{ or } X = \emptyset \ \&\ Y = \{z\} \textbf{ or } X = \{z\} \ \&\ Y = \emptyset,$$

(44) $$\{z\} = X \cup Y \ \&\ X \neq Y \textbf{ implies } X = \emptyset \textbf{ or } Y = \emptyset,$$

(45) $$\{x\} \cup X = X \textbf{ or } X \cup \{x\} = X \textbf{ implies } x \in X,$$

(46) $$x \in X \textbf{ implies } \{x\} \cup X = X \ \&\ X \cup \{x\} = X,$$

(47) $$\{x,y\} \cup Z = Z \textbf{ or } Z \cup \{x,y\} = Z \textbf{ implies } x \in Z \ \&\ y \in Z,$$

(48) $$x \in Z \ \&\ y \in Z \textbf{ implies } \{x,y\} \cup Z = Z \ \&\ Z \cup \{x,y\} = Z,$$

(49) $$\{x\} \cup X \neq \emptyset \ \&\ X \cup \{x\} \neq \emptyset,$$

(50) $$\{x,y\} \cup X \neq \emptyset \ \&\ X \cup \{x,y\} \neq \emptyset,$$

(51) $$X \cap \{x\} = \{x\} \textbf{ or } \{x\} \cap X = \{x\} \textbf{ implies } x \in X,$$

(52) $$x \in X \textbf{ implies } X \cap \{x\} = \{x\} \ \&\ \{x\} \cap X = \{x\},$$

(53) $$x \in Z \ \&\ y \in Z \textbf{ implies } \{x,y\} \cap Z = \{x,y\} \ \&\ \{x,y\} = Z \cap \{x,y\},$$

(54) $$\{x\} \cap X = \emptyset \textbf{ or } X \cap \{x\} = \emptyset \textbf{ implies not}\, x \in X,$$

(55) $$\{x,y\} \cap Z = \emptyset \textbf{ or } Z \cap \{x,y\} = \emptyset \textbf{ implies not}\, x \in Z \ \&\ \textbf{not}\, y \in Z,$$

(56) $$\textbf{not}\, x \in X \textbf{ implies } \{x\} \cap X = \emptyset \ \&\ X \cap \{x\} = \emptyset,$$

(57) $$\textbf{not}\, x \in Z \ \&\ \textbf{not}\, y \in Z \textbf{ implies } \{x,y\} \cap Z = \emptyset \ \&\ Z \cap \{x,y\} = \emptyset,$$

(58) $$\{x\} \cap X = \emptyset \textbf{ or } \{x\} \cap X = \{x\} \ \&\ X \cap \{x\} = \{x\},$$

(59) $$\{x,y\} \cap X = \{x\} \textbf{ or } X \cap \{x,y\} = \{x\} \textbf{ implies not}\, y \in X \textbf{ or } x = y,$$

$$x \in X \;\&\; (\textbf{not}\, y \in X \textbf{ or } x = y) \textbf{ implies } \{x,y\} \cap X = \{x\} \;\&\; X \cap \{x,y\} = \{x\}, \tag{60}$$

$$\{x,y\} \cap X = \{y\} \textbf{ or } X \cap \{x,y\} = \{y\} \textbf{ implies not}\, x \in X \textbf{ or } x = y, \tag{61}$$

$$y \in X \;\&\; (\textbf{not}\, x \in X \textbf{ or } x = y) \textbf{ implies } \{x,y\} \cap X = \{y\} \;\&\; X \cap \{x,y\} = \{y\}, \tag{62}$$

$$\{x,y\} \cap X = \{x,y\} \textbf{ or } X \cap \{x,y\} = \{x,y\} \textbf{ implies } x \in X \;\&\; y \in X, \tag{63}$$

$$z \in X \setminus \{x\} \textbf{ iff } z \in X \;\&\; z \neq x, \tag{64}$$

$$X \setminus \{x\} = X \textbf{ iff not}\, x \in X, \tag{65}$$

$$X \setminus \{x\} = \emptyset \textbf{ implies } X = \emptyset \textbf{ or } X = \{x\}, \tag{66}$$

$$\{x\} \setminus X = \{x\} \textbf{ iff not}\, x \in X, \tag{67}$$

$$\{x\} \setminus X = \emptyset \textbf{ iff } x \in X, \tag{68}$$

$$\{x\} \setminus X = \emptyset \textbf{ or } \{x\} \setminus X = \{x\}, \tag{69}$$

$$\{x,y\} \setminus X = \{x\} \textbf{ iff not}\, x \in X \;\&\; (y \in X \textbf{ or } x = y), \tag{70}$$

$$\{x,y\} \setminus X = \{y\} \textbf{ iff } (x \in X \textbf{ or } x = y) \;\&\; \textbf{not}\, y \in X, \tag{71}$$

$$\{x,y\} \setminus X = \{x,y\} \textbf{ iff not}\, x \in X \;\&\; \textbf{not}\, y \in X, \tag{72}$$

$$\{x,y\} \setminus X = \emptyset \textbf{ iff } x \in X \;\&\; y \in X, \tag{73}$$

$$\begin{gathered}\{x,y\} \setminus X = \emptyset \\ \textbf{or } \{x,y\} \setminus X = \{x\} \textbf{ or } \{x,y\} \setminus X = \{y\} \textbf{ or } \{x,y\} \setminus X = \{x,y\},\end{gathered} \tag{74}$$

$$X \setminus \{x,y\} = \emptyset \textbf{ iff } X = \emptyset \textbf{ or } X = \{x\} \textbf{ or } X = \{y\} \textbf{ or } X = \{x,y\}, \tag{75}$$

$$\emptyset \in \operatorname{bool} A, \tag{76}$$

$$A \in \operatorname{bool} A, \tag{77}$$

$$\operatorname{bool} A \neq \emptyset, \tag{78}$$

$$A \subseteq B \textbf{ implies } \operatorname{bool} A \subseteq \operatorname{bool} B, \tag{79}$$

$$\{A\} \subseteq \operatorname{bool} A, \tag{80}$$

$$\operatorname{bool} A \cup \operatorname{bool} B \subseteq \operatorname{bool}(A \cup B), \tag{81}$$

$$\operatorname{bool} A \cup \operatorname{bool} B = \operatorname{bool}(A \cup B) \textbf{ implies } A \subseteq B \textbf{ or } B \subseteq A, \tag{82}$$

$$\operatorname{bool}(A \cap B) = \operatorname{bool} A \cap \operatorname{bool} B, \tag{83}$$

$$\text{bool}(A \setminus B) \subseteq \{\emptyset\} \cup (\text{bool}\,A \setminus \text{bool}\,B), \tag{84}$$

$$X \in \text{bool}(A \setminus B) \textbf{ iff } X \subseteq A \;\&\; X \text{ misses } B, \tag{85}$$

$$\text{bool}(A \setminus B) \cup \text{bool}(B \setminus A) \subseteq \text{bool}(A \dot{-} B), \tag{86}$$

$$X \in \text{bool}(A \dot{-} B) \textbf{ iff } X \subseteq A \cup B \;\&\; X \text{ misses } A \cap B, \tag{87}$$

$$X \in \text{bool}\,A \;\&\; Y \in \text{bool}\,A \textbf{ implies } X \cup Y \in \text{bool}\,A, \tag{88}$$

$$X \in \text{bool}\,A \textbf{ or } Y \in \text{bool}\,A \textbf{ implies } X \cap Y \in \text{bool}\,A, \tag{89}$$

$$X \in \text{bool}\,A \textbf{ implies } X \setminus Y \in \text{bool}\,A, \tag{90}$$

$$X \in \text{bool}\,A \;\&\; Y \in \text{bool}\,A \textbf{ implies } X \dot{-} Y \in \text{bool}\,A, \tag{91}$$

$$X \in A \textbf{ implies } X \subseteq \bigcup A, \tag{92}$$

$$\bigcup \{X, Y\} = X \cup Y, \tag{93}$$

$$(\textbf{for } X \textbf{ st } X \in A \textbf{ holds } X \subseteq Z) \textbf{ implies } \bigcup A \subseteq Z, \tag{94}$$

$$A \subseteq B \textbf{ implies } \bigcup A \subseteq \bigcup B, \tag{95}$$

$$\bigcup (A \cup B) = \bigcup A \cup \bigcup B, \tag{96}$$

$$\bigcup (A \cap B) \subseteq \bigcup A \cap \bigcup B, \tag{97}$$

$$(\textbf{for } X \textbf{ st } X \in A \textbf{ holds } X \cap B = \emptyset) \textbf{ implies } \bigcup (A) \cap B = \emptyset, \tag{98}$$

$$\bigcup \text{bool}\,A = A, \tag{99}$$

$$A \subseteq \text{bool} \bigcup A, \tag{100}$$

$$\begin{gathered}(\textbf{for } X,Y \textbf{ st } X \neq Y \;\&\; X \in A \cup B \;\&\; Y \in A \cup B \textbf{ holds } X \cap Y = \emptyset) \\ \textbf{implies } \bigcup (A \cap B) = \bigcup A \cap \bigcup B,\end{gathered} \tag{101}$$

$$z \in [:X, Y:] \textbf{ implies ex } x,y \textbf{ st } \langle x, y \rangle = z, \tag{102}$$

$$A \subseteq [:X, Y:] \;\&\; z \in A \textbf{ implies ex } x,y \textbf{ st } x \in X \;\&\; y \in Y \;\&\; z = \langle x, y \rangle, \tag{103}$$

$$\begin{gathered} z \in [:X1,Y1:] \cap [:X2,Y2:] \\ \textbf{implies ex } x,y \textbf{ st } z = \langle x, y \rangle \;\&\; x \in X1 \cap X2 \;\&\; y \in Y1 \cap Y2,\end{gathered} \tag{104}$$

$$[:X, Y:] \subseteq \text{bool}\,\text{bool}(X \cup Y), \tag{105}$$

$$\langle x, y \rangle \in [:X, Y:] \textbf{ iff } x \in X \;\&\; y \in Y, \tag{106}$$

$$\langle x,y\rangle \in [\!:X,Y:\!]\ \mathbf{implies}\ \langle y,x\rangle \in [\!:Y,X:\!], \tag{107}$$

$$(\mathbf{for}\ x,y\ \mathbf{holds}\ \langle x,y\rangle \in [\!:X1,Y1:\!]\ \mathbf{iff}\ \langle x,y\rangle \in [\!:X2,Y2:\!])\ \mathbf{implies}\ [\!:X1,Y1:\!] = [\!:X2,Y2:\!], \tag{108}$$

$$A \subseteq [\!:X,Y:\!]\ \&\ (\mathbf{for}\ x,y\ \mathbf{st}\ \langle x,y\rangle \in A\ \mathbf{holds}\ \langle x,y\rangle \in B)\ \mathbf{implies}\ A \subseteq B, \tag{109}$$

$$A \subseteq [\!:X1,Y1:\!]\ \&\ B \subseteq [\!:X2,Y2:\!]\ \&\ (\mathbf{for}\ x,y\ \mathbf{holds}\ \langle x,y\rangle \in A\ \mathbf{iff}\ \langle x,y\rangle \in B)\ \mathbf{implies}\ A = B, \tag{110}$$

$$(\mathbf{for}\ z\ \mathbf{st}\ z \in A\ \mathbf{ex}\ x,y\ \mathbf{st}\ z = \langle x,y\rangle)\ \&\ (\mathbf{for}\ x,y\ \mathbf{st}\ \langle x,y\rangle \in A\ \mathbf{holds}\ \langle x,y\rangle \in B)\ \mathbf{implies}\ A \subseteq B, \tag{111}$$

$$(\mathbf{for}\ z\ \mathbf{st}\ z \in A\ \mathbf{ex}\ x,y\ \mathbf{st}\ z = \langle x,y\rangle)\ \&\ (\mathbf{for}\ z\ \mathbf{st}\ z \in B\ \mathbf{ex}\ x,y\ \mathbf{st}\ z = \langle x,y\rangle)\ \&\ (\mathbf{for}\ x,y\ \mathbf{holds}\ \langle x,y\rangle \in A\ \mathbf{iff}\ \langle x,y\rangle \in B)\ \mathbf{implies}\ A = B, \tag{112}$$

$$[\!:X,Y:\!] = \emptyset\ \mathbf{iff}\ X = \emptyset\ \mathbf{or}\ Y = \emptyset, \tag{113}$$

$$X \neq \emptyset\ \&\ Y \neq \emptyset\ \&\ [\!:X,Y:\!] = [\!:Y,X:\!]\ \mathbf{implies}\ X = Y, \tag{114}$$

$$[\!:X,X:\!] = [\!:Y,Y:\!]\ \mathbf{implies}\ X = Y, \tag{115}$$

$$X \subseteq [\!:X,X:\!]\ \mathbf{implies}\ X = \emptyset, \tag{116}$$

$$Z \neq \emptyset\ \&\ ([\!:X,Z:\!] \subseteq [\!:Y,Z:\!]\ \mathbf{or}\ [\!:Z,X:\!] \subseteq [\!:Z,Y:\!])\ \mathbf{implies}\ X \subseteq Y, \tag{117}$$

$$X \subseteq Y\ \mathbf{implies}\ [\!:X,Z:\!] \subseteq [\!:Y,Z:\!]\ \&\ [\!:Z,X:\!] \subseteq [\!:Z,Y:\!], \tag{118}$$

$$X1 \subseteq Y1\ \&\ X2 \subseteq Y2\ \mathbf{implies}\ [\!:X1,X2:\!] \subseteq [\!:Y1,Y2:\!], \tag{119}$$

$$[\!:X \cup Y,Z:\!] = [\!:X,Z:\!] \cup [\!:Y,Z:\!]\ \&\ [\!:Z,X \cup Y:\!] = [\!:Z,X:\!] \cup [\!:Z,Y:\!], \tag{120}$$

$$[\!:X1 \cup X2,Y1 \cup Y2:\!] = [\!:X1,Y1:\!] \cup [\!:X1,Y2:\!] \cup [\!:X2,Y1:\!] \cup [\!:X2,Y2:\!], \tag{121}$$

$$[\!:X \cap Y,Z:\!] = [\!:X,Z:\!] \cap [\!:Y,Z:\!]\ \&\ [\!:Z,X \cap Y:\!] = [\!:Z,X:\!] \cap [\!:Z,Y:\!], \tag{122}$$

$$[\!:X1 \cap X2,Y1 \cap Y2:\!] = [\!:X1,Y1:\!] \cap [\!:X2,Y2:\!], \tag{123}$$

$$A \subseteq X\ \&\ B \subseteq Y\ \mathbf{implies}\ [\!:A,Y:\!] \cap [\!:X,B:\!] = [\!:A,B:\!], \tag{124}$$

$$[\!:X \setminus Y,Z:\!] = [\!:X,Z:\!] \setminus [\!:Y,Z:\!]\ \&\ [\!:Z,X \setminus Y:\!] = [\!:Z,X:\!] \setminus [\!:Z,Y:\!], \tag{125}$$

$$[\!:X1,X2:\!] \setminus [\!:Y1,Y2:\!] = [\!:X1 \setminus Y1,X2:\!] \cup [\!:X1,X2 \setminus Y2:\!], \tag{126}$$

$$X1 \cap X2 = \emptyset\ \mathbf{or}\ Y1 \cap Y2 = \emptyset\ \mathbf{implies}\ [\!:X1,Y1:\!] \cap [\!:X2,Y2:\!] = \emptyset, \tag{127}$$

$$\langle x, y\rangle \in [:\{z\}, Y:] \textbf{ iff } x = z \ \& \ y \in Y, \tag{128}$$

$$\langle x, y\rangle \in [:X, \{z\}:] \textbf{ iff } x \in X \ \& \ y = z, \tag{129}$$

$$X \neq \emptyset \textbf{ implies } [:\{x\}, X:] \neq \emptyset \ \& \ [:X, \{x\}:] \neq \emptyset, \tag{130}$$

$$x \neq y \textbf{ implies } [:\{x\}, X:] \cap [:\{y\}, Y:] = \emptyset \ \& \ [:X, \{x\}:] \cap [:Y, \{y\}:] = \emptyset, \tag{131}$$

$$[:\{x, y\}, X:] = [:\{x\}, X:] \cup [:\{y\}, X:] \ \& \ [:X, \{x, y\}:] = [:X, \{x\}:] \cup [:X, \{y\}:], \tag{132}$$

$$Z = [:X, Y:] \textbf{ iff for } z \textbf{ holds } z \in Z \textbf{ iff ex } x, y \textbf{ st } x \in X \ \& \ y \in Y \ \& \ z = \langle x, y\rangle, \tag{133}$$

$$X1 \neq \emptyset \ \& \ Y1 \neq \emptyset \ \& \ [:X1, Y1:] = [:X2, Y2:] \textbf{ implies } X1 = X2 \ \& \ Y1 = Y2, \tag{134}$$

$$X \subseteq [:X, Y:] \textbf{ or } X \subseteq [:Y, X:] \textbf{ implies } X = \emptyset. \tag{135}$$

References

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[2] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1, 1990.

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