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Compact Spaces

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Summary. The article contains definition of a compact space and some theorems about compact spaces. The notions of a cover of a set and a centered family are defined in the article to be used in these theorems. A set is compact in the topological space if and only if every open cover of the set has a finite subcover. This definition is equivalent, what has been shown next, to the following definition: a set is compact if and only if a subspace generated by that set is compact. Some theorems about mappings and homeomorphisms of compact spaces have been also proved. The following schemes used in proofs of theorems have been proved in the article : FuncExChoice - the scheme of choice of a function, BiFuncEx - the scheme of parallel choice of two functions and the theorem about choice of a finite counter image of a finite image.

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The articles [6], [1], [4], [3], [5], and [2] provide the terminology and notation for this paper. We follow a convention: x, y, z are arbitrary, Y, Z denote sets, and f denotes a function. In the article we present several logical schemes. The scheme *FuncExChoice* deals with a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set and a binary predicate $\mathcal{P}$ and states that:

there exists f being a function such that $\operatorname{dom} f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the parameters satisfy the following condition:

- for every x such that $x \in \mathcal{A}$ there exists y such that $y \in \mathcal{B}$ and $\mathcal{P}[x, y]$.

The scheme *BiFuncEx* deals with a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set, a constant $\mathcal{C}$ that is a set and a ternary predicate $\mathcal{P}$ and states that:

there exist f, g being functions such that $\operatorname{dom} f = \mathcal{A}$ and $\operatorname{dom} g = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, f(x), g(x)]$

provided the parameters satisfy the following condition:

- if $x \in \mathcal{A}$, then there exist y, z such that $y \in \mathcal{B}$ and $z \in \mathcal{C}$ and $\mathcal{P}[x, y, z]$.

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Next we state a proposition

(1) If Z is finite and $Z \subseteq \operatorname{rng} f$, then there exists Y such that $Y \subseteq \operatorname{dom} f$ and Y is finite and $f^\circ Y = Z$.

For simplicity we adopt the following convention: T, S are topological spaces, A is a subspace of T, p, q are points of T, P, Q, W, V are subsets of T, and F, G are families of subsets of T. Let us consider T, F, P. The predicate F is a cover of P is defined by:

$P \subseteq \bigcup F$.

One can prove the following proposition

(2) F is a cover of P if and only if $P \subseteq \bigcup F$.

Let us consider T, F. The predicate F is centered is defined by:

$F \neq \emptyset$ and for every G such that $G \neq \emptyset$ and $G \subseteq F$ and G is finite holds $\bigcap G \neq \emptyset$.

One can prove the following proposition

(3) F is centered if and only if $F \neq \emptyset$ and for every G such that $G \neq \emptyset$ and $G \subseteq F$ and G is finite holds $\bigcap G \neq \emptyset$.

We now define five new predicates. Let us consider T. The predicate T is compact is defined by:

for every F such that F is a cover of T and F is open there exists G such that $G \subseteq F$ and G is a cover of T and G is finite.

The predicate T is a T2 space is defined by:

for all p, q such that $p \neq q$ there exist W, V such that W is open and V is open and $p \in W$ and $q \in V$ and $W \cap V = \emptyset$.

The predicate T is a T3 space is defined by:

for every p for every P such that $P \neq \emptyset$ and P is closed and $p \notin P$ there exist W, V such that W is open and V is open and $p \in W$ and $P \subseteq V$ and $W \cap V = \emptyset$.

The predicate T is a T4 space is defined by:

for all W, V such that $W \neq \emptyset$ and $V \neq \emptyset$ and W is closed and V is closed and $W \cap V = \emptyset$ there exist P, Q such that P is open and Q is open and $W \subseteq P$ and $V \subseteq Q$ and $P \cap Q = \emptyset$.

Let us consider P. The predicate P is compact is defined by:

for every F such that F is a cover of P and F is open there exists G such that $G \subseteq F$ and G is a cover of P and G is finite.

We now state a number of propositions:

(4) T is compact if and only if for every F such that F is a cover of T and F is open there exists G such that $G \subseteq F$ and G is a cover of T and G is finite.

(5) T is a T2 space if and only if for all p, q such that $p \neq q$ there exist W, V such that W is open and V is open and $p \in W$ and $q \in V$ and $W \cap V = \emptyset$.

(6) T is a T3 space if and only if for every p for every P such that $P \neq \emptyset$ and P is closed and $p \notin P$ there exist W, V such that W is open and V is open and $p \in W$ and $P \subseteq V$ and $W \cap V = \emptyset$.

(7) T is a T4 space if and only if for all P, Q such that $P \neq \emptyset$ and $Q \neq \emptyset$ and P is closed and Q is closed and $P \cap Q = \emptyset$ there exist W, V such that W is open and V is open and $P \subseteq W$ and $Q \subseteq V$ and $W \cap V = \emptyset$.

(8) P is compact if and only if for every F such that F is a cover of P and F is open there exists G such that $G \subseteq F$ and G is a cover of P and G is finite.

(9) $\emptyset_T$ is compact.

(10) T is compact if and only if Ω_T is compact.

(11) If $Q \subseteq \Omega_A$, then Q is compact if and only if for every subset P of A such that $P = Q$ holds P is compact.

(12) If $P \neq \emptyset$, then P is compact if and only if $T \restriction P$ is compact.

(13) T is compact if and only if for every F such that F is centered and F is closed holds $\bigcap F \neq \emptyset$.

(14) T is compact if and only if for every F such that $F \neq \emptyset$ and F is closed and $\bigcap F = \emptyset$ there exists G such that $G \neq \emptyset$ and $G \subseteq F$ and G is finite and $\bigcap G = \emptyset$.

(15) For every T such that T is a T2 space for every subset A of T such that $A \neq \emptyset$ and A is compact for every p such that $p \notin A$ there exist P, Q such that P is open and Q is open and $p \in P$ and $A \subseteq Q$ and $P \cap Q = \emptyset$.

(16) If T is a T2 space and P is compact, then P is closed.

(17) If T is compact and P is closed, then P is compact.

(18) If P is compact and $Q \subseteq P$ and Q is closed, then Q is compact.

(19) If P is compact and Q is compact, then $P \cup Q$ is compact.

(20) If T is a T2 space and P is compact and Q is compact, then $P \cap Q$ is compact.

(21) If T is a T2 space and T is compact, then T is a T3 space.

(22) If T is a T2 space and T is compact, then T is a T4 space.

In the sequel f will be a map from T into S. Next we state four propositions:

(23) If T is compact and f is continuous and $\operatorname{rng} f = \Omega_S$, then S is compact.

(24) If f is continuous and $\operatorname{rng} f = \Omega_S$ and P is compact, then $f \circ P$ is compact.

(25) If T is compact and S is a T2 space and $\operatorname{rng} f = \Omega_S$ and f is continuous, then for every P such that P is closed holds $f \circ P$ is closed.

(26) If T is compact and S is a T2 space and $\operatorname{rng} f = \Omega_S$ and f is one-to-one and f is continuous, then f is a homeomorphism.

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