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Partially Ordered Sets

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Summary. In the beginning of this article we define the choice function of a non-empty set family that does not contain $\emptyset$ as introduced in [5, pages 88–89]. We define order of a set as a relation being reflexive, antisymmetric and transitive in the set, partially ordered set as structure non-emty set and order of the set, chains, lower and upper cone of a subset, initial segments of element and subset of partially ordered set. Some theorems that belong rather to [4] or [9] are proved.

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The notation and terminology used in this paper have been introduced in the following articles: [6], [2], [3], [7], [9], [8], and [1]. We adopt the following convention: X, Y will denote sets, x, y, y_1, y_2, z will be arbitrary, and f will denote a function. In the article we present several logical schemes. The scheme *FuncExS* deals with a constant $\mathcal{A}$ that is a set and a binary predicate $\mathcal{P}$ and states that:

there exists f such that $\operatorname{dom} f = \mathcal{A}$ and for every X such that $X \in \mathcal{A}$ holds $\mathcal{P}[X, f(X)]$

provided the parameters satisfy the following conditions:

- for all X, y_1, y_2 such that $X \in \mathcal{A}$ and $\mathcal{P}[X, y_1]$ and $\mathcal{P}[X, y_2]$ holds $y_1 = y_2$,
- for every X such that $X \in \mathcal{A}$ there exists y such that $\mathcal{P}[X, y]$.

The scheme *LambdaS* concerns a constant $\mathcal{A}$ that is a set and a unary functor $\mathcal{F}$ and states that:

there exists f such that $\operatorname{dom} f = \mathcal{A}$ and for every X such that $X \in \mathcal{A}$ holds $f(X) = \mathcal{F}(X)$

for all values of the parameters.

In the sequel M will be a non-empty family of sets and F will be a function from M into $\bigcup M$. Let us consider M. Let us assume that $\emptyset \notin M$. The mode choice function of M, which widens to the type a function from M into $\bigcup M$, is defined by:

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for every X such that $X \in M$ holds it$(X) \in X$.

The following proposition is true

(1) If $\emptyset \notin M$ and for every X such that $X \in M$ holds $F(X) \in X$, then F is a choice function of M.

In the sequel CF will denote a choice function of M. Next we state a proposition

(2) If $\emptyset \notin M$, then for every X such that $X \in M$ holds $CF(X) \in X$.

In the sequel D, D_1 will denote non-empty sets. Let us consider D. The functor 2^D_+ yielding a non-empty family of sets, is defined by:

$2^D_+ = 2^D \setminus \{\emptyset\}$.

Next we state several propositions:

(3) $2^D_+ = 2^D \setminus \{\emptyset\}$.

(4) $\emptyset \notin 2^D_+$.

(5) $D_1 \subseteq D$ if and only if $D_1 \in 2^D_+$.

(6) D_1 is a subset of D if and only if $D_1 \in 2^D_+$.

(7) $D \in 2^D_+$.

In the sequel P denotes a relation and R denotes a relation on X. Let us consider X. The mode order in X, which widens to the type a relation on X, is defined by:

it is reflexive in X and it is antisymmetric in X and it is transitive in X.

We now state a proposition

(8) If R is reflexive in X and R is antisymmetric in X and R is transitive in X, then R is an order in X.

In the sequel O denotes an order in X. We now state several propositions:

(9) O is reflexive in X.

(10) O is antisymmetric in X.

(11) O is transitive in X.

(12) If $x \in X$, then $\langle x, x \rangle \in O$.

(13) If $x \in X$ and $y \in X$ and $\langle x, y \rangle \in O$ and $\langle y, x \rangle \in O$, then $x = y$.

(14) If $x \in X$ and $y \in X$ and $z \in X$ and $\langle x, y \rangle \in O$ and $\langle y, z \rangle \in O$, then $\langle x, z \rangle \in O$.

We consider posets which are systems

$\langle$ a carrier, an order $\rangle$

where the carrier is a non-empty set and the order is an order in the carrier. In the sequel A will denote a poset. Let us consider A. An element of A is an element of the carrier of A.

Let us consider A. A subset of A is a subset of the carrier of A.

In the sequel a is an element of the carrier of A and S is a subset of the carrier of A. One can prove the following propositions:

(15) a is an element of A.

(16) S is a subset of A.

(17) $x \in$the carrier of A if and only if x is an element of A.

(18) $X \subseteq$the carrier of A if and only if X is a subset of A.

(19) If $x \in S$, then x is an element of A.

We follow the rules: a, a_1, a_2, a_3, b, c denote elements of A and S, T denote subsets of A. Let us consider A, a. Then $\{a\}$ is a subset of A.

Let us consider A, a_1, a_2. Then $\{a_1, a_2\}$ is a subset of A.

Let us consider A, S, T. Then $S \cup T$ is a subset of A. Then $S \cap T$ is a subset of A. Then $S \setminus T$ is a subset of A. Then $S \dot{-} T$ is a subset of A.

Let us consider A. The functor $\emptyset_A$ yielding a subset of A, is defined by:
$\emptyset_A = \emptyset$.

Let us consider A. The functor Ω_A yielding a subset of A, is defined by:
$\Omega_A =$the carrier of A.

One can prove the following propositions:

(20) $\emptyset_A = \emptyset$.

(21) $\Omega_A =$the carrier of A.

Let us consider A, a_1, a_2. The predicate $a_1 \leq a_2$ is defined by:
$\langle a_1, a_2 \rangle \in$the order of A.

Let us consider A, a_1, a_2. The predicate $a_1 < a_2$ is defined by:
$a_1 \leq a_2$ and $a_1 \neq a_2$.

One can prove the following propositions:

(22) $a_1 \leq a_2$ if and only if $\langle a_1, a_2 \rangle \in$the order of A.

(23) $a_1 < a_2$ if and only if $a_1 \leq a_2$ and $a_1 \neq a_2$.

(24) $a \leq a$.

(25) If $a_1 \leq a_2$ and $a_2 \leq a_1$, then $a_1 = a_2$.

(26) If $a_1 \leq a_2$ and $a_2 \leq a_3$, then $a_1 \leq a_3$.

(27) $a \not< a$.

(28) this conjunction is not true: $a_1 < a_2$ and $a_2 < a_1$.

(29) If $a_1 < a_2$ and $a_2 < a_3$, then $a_1 < a_3$.

(30) If $a_1 \leq a_2$, then $a_2 \not< a_1$.

(31) If $a_1 < a_2$, then $a_2 \not\leq a_1$.

(32) If $a_1 < a_2$ and $a_2 \leq a_3$ or $a_1 \leq a_2$ and $a_2 < a_3$, then $a_1 < a_3$.

Let us consider A. The mode chain of A, which widens to the type a subset of A, is defined by:
the order of A is strongly connected in it .

One can prove the following proposition

(33) If the order of A is strongly connected in S, then S is a chain of A.

In the sequel C will denote a chain of A. One can prove the following propositions:

(34) the order of A is strongly connected in C.

(35) $\{a\}$ is a chain of A.

(36) $\{a_1, a_2\}$ is a chain of A if and only if $a_1 \leq a_2$ or $a_2 \leq a_1$.

(37) If $S \subseteq C$, then S is a chain of A.

(38) There exists C such that $a_1 \in C$ and $a_2 \in C$ if and only if $a_1 \leq a_2$ or $a_2 \leq a_1$.

(39) There exists C such that $a_1 \in C$ and $a_2 \in C$ if and only if $a_1 < a_2$ if and only if $a_2 \not\leq a_1$.

(40) If the order of A well orders T, then T is a chain of A.

Let us consider A, S. The functor $\mathrm{UpperCone}\, S$ yields a subset of A and is defined by:

$\mathrm{UpperCone}\, S = \{a_1 : \bigvee_{a_2}[a_2 \in S \Rightarrow a_2 < a_1]\}$.

Let us consider A, S. The functor $\mathrm{LowerCone}\, S$ yielding a subset of A, is defined by:

$\mathrm{LowerCone}\, S = \{a_1 : \bigvee_{a_2}[a_2 \in S \Rightarrow a_1 < a_2]\}$.

The following propositions are true:

(41) $\mathrm{UpperCone}\, S = \{a_1 : \bigvee_{a_2}[a_2 \in S \Rightarrow a_2 < a_1]\}$.

(42) $\mathrm{LowerCone}\, S = \{a_1 : \bigvee_{a_2}[a_2 \in S \Rightarrow a_1 < a_2]\}$.

(43) $\mathrm{UpperCone}\, \emptyset_A =$the carrier of A.

(44) $\mathrm{UpperCone}\, \Omega_A = \emptyset$.

(45) $\mathrm{LowerCone}\, \emptyset_A =$the carrier of A.

(46) $\mathrm{LowerCone}\, \Omega_A = \emptyset$.

(47) If $a \in S$, then $a \notin \mathrm{UpperCone}\, S$.

(48) $a \notin \mathrm{UpperCone}\{a\}$.

(49) If $a \in S$, then $a \notin \mathrm{LowerCone}\, S$.

(50) $a \notin \mathrm{LowerCone}\{a\}$.

(51) $c < a$ if and only if $a \in \mathrm{UpperCone}\{c\}$.

(52) $a < c$ if and only if $a \in \mathrm{LowerCone}\{c\}$.

Let us consider A, S, a. The functor $\mathrm{InitSegm}(S, a)$ yields a subset of A and is defined by:

$\mathrm{InitSegm}(S, a) = \mathrm{LowerCone}\{a\} \cap S$.

Let us consider A, S. The mode initial segment of S, which widens to the type a subset of A, is defined by:

there exists a such that $a \in S$ and it $= \mathrm{InitSegm}(S, a)$ if $S \neq \emptyset$, it $= \emptyset$, otherwise.

The following propositions are true:

(53) $\mathrm{InitSegm}(S, a) = \mathrm{LowerCone}\{a\} \cap S$.

(54) If $S \neq \emptyset$ and there exists a such that $a \in S$ and $T = \mathrm{InitSegm}(S, a)$, then T is an initial segment of S.

(55) If $S = \emptyset$, then T is an initial segment of S if and only if $T = \emptyset$.

In the sequel I will be an initial segment of S and I_0 will be an initial segment of $\emptyset_A$. One can prove the following propositions:

(56) $x \in \mathrm{InitSegm}(S, a)$ if and only if $x \in \mathrm{LowerCone}\{a\}$ and $x \in S$.

(57) $a \in \mathrm{InitSegm}(S, b)$ if and only if $a < b$ and $a \in S$.

(58) If $S \neq \emptyset$, then there exists a such that $a \in S$ and $I = \mathrm{InitSegm}(S, a)$.

(59) If $a \in T$ and $S = \mathrm{InitSegm}(T, a)$, then S is an initial segment of T.

(60) $\mathrm{InitSegm}(\emptyset_A, a) = \emptyset$.

(61) $\mathrm{InitSegm}(S, a) \subseteq S$.

(62) $a \notin \mathrm{InitSegm}(S, a)$.

(63) $a_1 \in S$ and $a_1 < a_2$ if and only if $a_1 \in \mathrm{InitSegm}(S, a_2)$.

(64) If $a_1 < a_2$, then $\mathrm{InitSegm}(S, a_1) \subseteq \mathrm{InitSegm}(S, a_2)$.

(65) If $S \subseteq T$, then $\mathrm{InitSegm}(S, a) \subseteq \mathrm{InitSegm}(T, a)$.

(66) $I_0 = \emptyset$.

(67) $I \subseteq S$.

(68) $S \neq \emptyset$ if and only if S is not an initial segment of S.

(69) If $S \neq \emptyset$ or $T \neq \emptyset$ but S is an initial segment of T, then T is not an initial segment of S.

(70) If $a_1 < a_2$ and $a_1 \in S$ and $a_2 \in T$ and T is an initial segment of S, then $a_1 \in T$.

(71) If $a \in S$ and S is an initial segment of T, then $\mathrm{InitSegm}(S, a) = \mathrm{InitSegm}(T, a)$.

(72) If $S \subseteq T$ and the order of A well orders T and for all a_1, a_2 such that $a_2 \in S$ and $a_1 < a_2$ holds $a_1 \in S$, then $S = T$ or S is an initial segment of T.

(73) If $S \subseteq T$ and the order of A well orders T and for all a_1, a_2 such that $a_2 \in S$ and $a_1 \in T$ and $a_1 < a_2$ holds $a_1 \in S$, then $S = T$ or S is an initial segment of T.

In the sequel f will denote a choice function of $2_+^{\text{the carrier of } A}$. Let us consider A, f. The mode chain of f, which widens to the type a chain of A, is defined by:

it $\neq \emptyset$ and the order of A well orders it and for every a such that $a \in$ it holds $f(\mathrm{UpperCone}\,\mathrm{InitSegm}(\mathrm{it}, a)) = a$.

Next we state a proposition

(74) If $C \neq \emptyset$ and the order of A well orders C and for every a such that $a \in C$ holds $f(\mathrm{UpperCone}\,\mathrm{InitSegm}(C, a)) = a$, then C is a chain of f.

In the sequel fC, fC_1, fC_2 denote chains of f. Next we state a number of propositions:

(75) $fC \neq \emptyset$.

(76) the order of A well orders fC.

(77) If $a \in fC$, then $f(\mathrm{UpperCone}\,\mathrm{InitSegm}(fC, a)) = a$.

(78) $\{f(\text{the carrier of } A)\}$ is a chain of f.

(79) $f(\text{the carrier of } A) \in fC$.

(80) If $a \in fC$ and $b = f(\text{the carrier of } A)$, then $b \leq a$.

(81) If $a = f(\text{the carrier of } A)$, then $\mathrm{InitSegm}(fC, a) = \emptyset$.

(82) $fC_1 \cap fC_2 \neq \emptyset$.

(83) If $f C_1 \neq f C_2$, then $f C_1$ is an initial segment of $f C_2$ if and only if $f C_2$ is not an initial segment of $f C_1$.

(84) $f C_1 \neq f C_2$ and $f C_1 \subseteq f C_2$ if and only if $f C_1$ is an initial segment of $f C_2$.

Let us consider A, f. The functor $\mathrm{Chains}\, f$ yielding a non-empty set, is defined by:

$x \in \mathrm{Chains}\, f$ if and only if x is a chain of f.

One can prove the following propositions:

(85) If for every x holds $x \in D$ if and only if x is a chain of f, then $D = \mathrm{Chains}\, f$.

(86) $x \in \mathrm{Chains}\, f$ if and only if x is a chain of f.

(87) $\bigcup(\mathrm{Chains}\, f) \neq \emptyset$.

(88) If $f C \neq \bigcup(\mathrm{Chains}\, f)$ and $S = \bigcup(\mathrm{Chains}\, f)$, then $f C$ is an initial segment of S.

(89) $\bigcup(\mathrm{Chains}\, f)$ is a chain of f.

(90) $x \in X$ if and only if $\{x\} \in 2^X$.

(91) There exists X such that $X \neq \emptyset$ and $X \in Y$ if and only if $\bigcup Y \neq \emptyset$.

(92) P is strongly connected in X if and only if P is reflexive in X and P is connected in X.

(93) If P is reflexive in X and $Y \subseteq X$, then P is reflexive in Y.

(94) If P is antisymmetric in X and $Y \subseteq X$, then P is antisymmetric in Y.

(95) If P is transitive in X and $Y \subseteq X$, then P is transitive in Y.

(96) If P is strongly connected in X and $Y \subseteq X$, then P is strongly connected in Y.

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