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Sequences of Ordinal Numbers

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Summary. In the first part of the article we introduce the following operations: On X that yields the set of all ordinals which belong to the set X, Lim X that yields the set of all limit ordinals which belong to X, and inf X and sup X that yield the minimal ordinal belonging to X and the minimal ordinal greater than all ordinals belonging to X, respectively. The second part of the article starts with schemes that can be used to justify the correctness of definitions based on the transfinite induction (see [1] or [3]). The schemes are used to define addition, product and power of ordinal numbers. The operations of limes inferior and limes superior of sequences of ordinals are defined and the concepts of limet of ordinal sequence and increasing and continuous sequence are introduced.

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The papers [5], [2], [1], and [4] provide the terminology and notation for this paper. For simplicity we adopt the following rules: A, A_1, A_2, B, C, D will denote ordinal numbers, X, Y will denote sets, x, y will be arbitrary, and L, L_1 will denote transfinite sequences. The scheme *Ordinal_Ind* concerns a unary predicate $\mathcal{P}$ and states that:

for every A holds $\mathcal{P}[A]$

provided the parameter satisfies the following conditions:

- $\mathcal{P}[\mathbf{0}]$,
- for every A such that $\mathcal{P}[A]$ holds $\mathcal{P}[\operatorname{succ} A]$,
- for every A such that $A \neq \mathbf{0}$ and A is a limit ordinal number and for every B such that $B \in A$ holds $\mathcal{P}[B]$ holds $\mathcal{P}[A]$.

We now state several propositions:

(1) If $A \subseteq B$, then $\operatorname{succ} A \subseteq \operatorname{succ} B$.

(2) $\bigcup(\operatorname{succ} A) = A$.

(3) $\operatorname{succ} A \subseteq 2^A$.

(4) $\mathbf{0}$ is a limit ordinal number.

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(5) $\bigcup A \subseteq A$.

Let us consider L. The functor last L yielding a set, is defined by:
$\operatorname{last} L = L(\bigcup(\operatorname{dom} L))$.

Next we state two propositions:

(6) $\operatorname{last} L = L(\bigcup(\operatorname{dom} L))$.

(7) If $\operatorname{dom} L = \operatorname{succ} A$, then $\operatorname{last} L = L(A)$.

We now define two new functors. Let us consider X. The functor On X yields a set and is defined by:

$x \in \operatorname{On} X$ if and only if $x \in X$ and x is an ordinal number.

The functor Lim X yielding a set, is defined by:

$x \in \operatorname{Lim} X$ if and only if $x \in X$ and there exists A such that $x = A$ and A is a limit ordinal number.

Next we state a number of propositions:

(8) $x \in \operatorname{On} X$ if and only if $x \in X$ and x is an ordinal number.

(9) $\operatorname{On} X \subseteq X$.

(10) $\operatorname{On} A = A$.

(11) If $X \subseteq Y$, then $\operatorname{On} X \subseteq \operatorname{On} Y$.

(12) $x \in \operatorname{Lim} X$ if and only if $x \in X$ and there exists A such that $x = A$ and A is a limit ordinal number.

(13) $\operatorname{Lim} X \subseteq X$.

(14) If $X \subseteq Y$, then $\operatorname{Lim} X \subseteq \operatorname{Lim} Y$.

(15) $\operatorname{Lim} X \subseteq \operatorname{On} X$.

(16) For every D there exists A such that $D \in A$ and A is a limit ordinal number.

(17) If for every x such that $x \in X$ holds x is an ordinal number, then $\bigcap X$ is an ordinal number.

We now define four new functors. The constant $\mathbf{1}$ is an ordinal number and is defined by:

$\mathbf{1} = \operatorname{succ} \mathbf{0}$.

The constant ω is an ordinal number and is defined by:

$\mathbf{0} \in \omega$ and ω is a limit ordinal number and for every A such that $\mathbf{0} \in A$ and A is a limit ordinal number holds $\omega \subseteq A$.

Let us consider X. The functor inf X yields an ordinal number and is defined by:

$\inf X = \bigcap(\operatorname{On} X)$.

The functor sup X yielding an ordinal number, is defined by:

$\operatorname{On} X \subseteq \sup X$ and for every A such that $\operatorname{On} X \subseteq A$ holds $\sup X \subseteq A$.

We now state a number of propositions:

(18) $\mathbf{1} = \operatorname{succ} \mathbf{0}$.

(19) $\mathbf{0} \in \omega$ and ω is a limit ordinal number and for every A such that $\mathbf{0} \in A$ and A is a limit ordinal number holds $\omega \subseteq A$.

(20) $\inf X = \bigcap(\operatorname{On} X)$.

(21) $B = \sup X$ if and only if $\mathrm{On}\, X \subseteq B$ and for every A such that $\mathrm{On}\, X \subseteq A$ holds $B \subseteq A$.

(22) If $A \in X$, then $\inf X \subseteq A$.

(23) If $\mathrm{On}\, X \neq \emptyset$ and for every A such that $A \in X$ holds $D \subseteq A$, then $D \subseteq \inf X$.

(24) If $A \in X$ and $X \subseteq Y$, then $\inf Y \subseteq \inf X$.

(25) If $A \in X$, then $\inf X \in X$.

(26) $\sup A = A$.

(27) If $A \in X$, then $A \in \sup X$.

(28) If for every A such that $A \in X$ holds $A \in D$, then $\sup X \subseteq D$.

(29) If $A \in \sup X$, then there exists B such that $B \in X$ and $A \subseteq B$.

(30) If $X \subseteq Y$, then $\sup X \subseteq \sup Y$.

(31) $\sup\{A\} = \mathrm{succ}\, A$.

(32) $\inf X \subseteq \sup X$.

The scheme *TS_Lambda* concerns a constant $\mathcal{A}$ that is an ordinal number and a unary functor $\mathcal{F}$ and states that:

there exists L such that $\mathrm{dom}\, L = \mathcal{A}$ and for every A such that $A \in \mathcal{A}$ holds $L(A) = \mathcal{F}(A)$

for all values of the parameters.

The mode sequence of ordinal numbers, which widens to the type a transfinite sequence, is defined by:

there exists A such that $\mathrm{rng}\, \mathrm{it} \subseteq A$.

The following proposition is true

(33) L is a sequence of ordinal numbers if and only if there exists A such that $\mathrm{rng}\, L \subseteq A$.

Let us consider A. We see that it makes sense to consider the following mode for restricted scopes of arguments. Then all the objests of the mode transfinite sequence of elements of A are a sequence of ordinal numbers.

The arguments of the notions defined below are the following: L which is a sequence of ordinal numbers; A which is an object of the type reserved above. Then $L \restriction A$ is a sequence of ordinal numbers. Then $L(A)$ is a set.

In the sequel fi, *psi* are sequences of ordinal numbers. Next we state a proposition

(34) If $A \in \mathrm{dom}\, fi$, then $fi(A)$ is an ordinal number.

Now we present a number of schemes. The scheme *OS_Lambda* concerns a constant $\mathcal{A}$ that is an ordinal number and a unary functor $\mathcal{F}$ yielding an ordinal number and states that:

there exists fi such that $\mathrm{dom}\, fi = \mathcal{A}$ and for every A such that $A \in \mathcal{A}$ holds $fi(A) = \mathcal{F}(A)$

for all values of the parameters.

The scheme *TS_Uniq1* deals with a constant $\mathcal{A}$ that is an ordinal number, a constant $\mathcal{B}$, a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, a constant $\mathcal{C}$ that is a transfinite sequence and a constant $\mathcal{D}$ that is a transfinite sequence, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters satisfy the following conditions:

- $\operatorname{dom} \mathcal{C} = \mathcal{A}$,
- if $\mathbf{0} \in \mathcal{A}$, then $\mathcal{C}(\mathbf{0}) = \mathcal{B}$,
- for all A, x such that $\operatorname{succ} A \in \mathcal{A}$ and $x = \mathcal{C}(A)$ holds $\mathcal{C}(\operatorname{succ} A) = \mathcal{F}(A, x)$,
- for all A, L such that $A \in \mathcal{A}$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $L = \mathcal{C} \restriction A$ holds $\mathcal{C}(A) = \mathcal{G}(A, L)$,
- $\operatorname{dom} \mathcal{D} = \mathcal{A}$,
- if $\mathbf{0} \in \mathcal{A}$, then $\mathcal{D}(\mathbf{0}) = \mathcal{B}$,
- for all A, x such that $\operatorname{succ} A \in \mathcal{A}$ and $x = \mathcal{D}(A)$ holds $\mathcal{D}(\operatorname{succ} A) = \mathcal{F}(A, x)$,
- for all A, L such that $A \in \mathcal{A}$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $L = \mathcal{D} \restriction A$ holds $\mathcal{D}(A) = \mathcal{G}(A, L)$.

The scheme *TS_Exist1* concerns a constant $\mathcal{A}$ that is an ordinal number, a constant $\mathcal{B}$, a binary functor $\mathcal{F}$ and a binary functor $\mathcal{G}$ and states that:

there exists L such that $\operatorname{dom} L = \mathcal{A}$ but if $\mathbf{0} \in \mathcal{A}$, then $L(\mathbf{0}) = \mathcal{B}$ and for all A, x such that $\operatorname{succ} A \in \mathcal{A}$ and $x = L(A)$ holds $L(\operatorname{succ} A) = \mathcal{F}(A, x)$ and for all A, L_1 such that $A \in \mathcal{A}$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $L_1 = L \restriction A$ holds $L(A) = \mathcal{G}(A, L_1)$.

for all values of the parameters.

The scheme *TS_Result* deals with a constant $\mathcal{A}$ that is a transfinite sequence, a unary functor $\mathcal{F}$, a constant $\mathcal{B}$ that is an ordinal number, a constant $\mathcal{C}$, a binary functor $\mathcal{G}$ and a binary functor $\mathcal{H}$ and states that:

for every A such that $A \in \operatorname{dom} \mathcal{A}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$

provided the parameters satisfy the following conditions:

- Given A, x. Then $x = \mathcal{F}(A)$ if and only if there exists L such that $x = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\mathbf{0}) = \mathcal{C}$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} A$ and $y = L(C)$ holds $L(\operatorname{succ} C) = \mathcal{G}(C, y)$ and for all C, L_1 such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathcal{H}(C, L_1)$.
- $\operatorname{dom} \mathcal{A} = \mathcal{B}$,
- if $\mathbf{0} \in \mathcal{B}$, then $\mathcal{A}(\mathbf{0}) = \mathcal{C}$,
- for all A, y such that $\operatorname{succ} A \in \mathcal{B}$ and $y = \mathcal{A}(A)$ holds $\mathcal{A}(\operatorname{succ} A) = \mathcal{G}(A, y)$,
- for all A, L_1 such that $A \in \mathcal{B}$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $L_1 = \mathcal{A} \restriction A$ holds $\mathcal{A}(A) = \mathcal{H}(A, L_1)$.

The scheme *TS_Def* deals with a constant $\mathcal{A}$ that is an ordinal number, a constant $\mathcal{B}$, a binary functor $\mathcal{F}$ and a binary functor $\mathcal{G}$ and states that:

(i) there exist x, L such that $x = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} \mathcal{A}$ and $L(\mathbf{0}) = \mathcal{B}$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ and $y = L(C)$ holds $L(\operatorname{succ} C) = \mathcal{F}(C, y)$ and for all C, L_1 such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathcal{G}(C, L_1)$,

(ii) for arbitrary x_1, x_2 such that there exists L such that $x_1 = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} \mathcal{A}$ and $L(\mathbf{0}) = \mathcal{B}$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ and $y = L(C)$ holds $L(\operatorname{succ} C) = \mathcal{F}(C, y)$ and for all C, L_1 such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathcal{G}(C, L_1)$ and there exists L such that $x_2 = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} \mathcal{A}$ and $L(\mathbf{0}) = \mathcal{B}$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ and $y = L(C)$ holds $L(\operatorname{succ} C) = \mathcal{F}(C, y)$ and for all C, L_1 such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathcal{G}(C, L_1)$ holds $x_1 = x_2$.

for all values of the parameters.

The scheme *TS_Result0* deals with a unary functor $\mathcal{F}$, a constant $\mathcal{A}$, a binary functor $\mathcal{G}$ and a binary functor $\mathcal{H}$ and states that:

$\mathcal{F}(\mathbf{0}) = \mathcal{A}$

provided the parameters satisfy the following condition:

- Given A, x. Then $x = \mathcal{F}(A)$ if and only if there exists L such that $x = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\mathbf{0}) = \mathcal{A}$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} A$ and $y = L(C)$ holds $L(\operatorname{succ} C) = \mathcal{G}(C, y)$ and for all C, L_1 such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathcal{H}(C, L_1)$.

The scheme *TS_ResultS* deals with a constant $\mathcal{A}$, a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$ and a unary functor $\mathcal{H}$ and states that:

for every A holds $\mathcal{H}(\operatorname{succ} A) = \mathcal{F}(A, \mathcal{H}(A))$

provided the parameters satisfy the following condition:

- Given A, x. Then $x = \mathcal{H}(A)$ if and only if there exists L such that $x = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\mathbf{0}) = \mathcal{A}$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} A$ and $y = L(C)$ holds $L(\operatorname{succ} C) = \mathcal{F}(C, y)$ and for all C, L_1 such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathcal{G}(C, L_1)$.

The scheme *TS_ResultL* concerns a constant $\mathcal{A}$ that is a transfinite sequence, a constant $\mathcal{B}$ that is an ordinal number, a unary functor $\mathcal{F}$, a constant $\mathcal{C}$, a binary functor $\mathcal{G}$ and a binary functor $\mathcal{H}$ and states that:

$\mathcal{F}(\mathcal{B}) = \mathcal{H}(\mathcal{B}, \mathcal{A})$

provided the parameters satisfy the following conditions:

- Given A, x. Then $x = \mathcal{F}(A)$ if and only if there exists L such that $x = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\mathbf{0}) = \mathcal{C}$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} A$ and $y = L(C)$ holds $L(\operatorname{succ} C) = \mathcal{G}(C, y)$ and for all C, L_1 such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathcal{H}(C, L_1)$.
- $\mathcal{B} \neq \mathbf{0}$ and $\mathcal{B}$ is a limit ordinal number,
- $\operatorname{dom} \mathcal{A} = \mathcal{B}$,
- for every A such that $A \in \mathcal{B}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$.

The scheme *OS_Exist* concerns a constant $\mathcal{A}$ that is an ordinal number, a constant $\mathcal{B}$ that is an ordinal number, a binary functor $\mathcal{F}$ yielding an ordinal number and a binary functor $\mathcal{G}$ yielding an ordinal number and states that:

there exists fi such that $\operatorname{dom} fi = \mathcal{A}$ but if $\mathbf{0} \in \mathcal{A}$, then $fi(\mathbf{0}) = \mathcal{B}$ and for all A, B such that $\operatorname{succ} A \in \mathcal{A}$ and $B = fi(A)$ holds $fi(\operatorname{succ} A) = \mathcal{F}(A, B)$ and

for all A, *psi* such that $A \in \mathcal{A}$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $psi = fi \restriction A$ holds $fi(A) = \mathcal{G}(A, psi)$.
for all values of the parameters.

The scheme *OS_Result* deals with a constant $\mathcal{A}$ that is a sequence of ordinal numbers, a unary functor $\mathcal{F}$ yielding an ordinal number, a constant $\mathcal{B}$ that is an ordinal number, a constant $\mathcal{C}$ that is an ordinal number, a binary functor $\mathcal{G}$ yielding an ordinal number and a binary functor $\mathcal{H}$ yielding an ordinal number and states that:

for every A such that $A \in \operatorname{dom} \mathcal{A}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$

provided the parameters satisfy the following conditions:

- Given A, B. Then $B = \mathcal{F}(A)$ if and only if there exists fi such that $B = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} A$ and $fi(\mathbf{0}) = \mathcal{C}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} A$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = \mathcal{G}(C, D)$ and for all C, *psi* such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{H}(C, psi)$.
- $\operatorname{dom} \mathcal{A} = \mathcal{B}$,
- if $\mathbf{0} \in \mathcal{B}$, then $\mathcal{A}(\mathbf{0}) = \mathcal{C}$,
- for all A, B such that $\operatorname{succ} A \in \mathcal{B}$ and $B = \mathcal{A}(A)$ holds $\mathcal{A}(\operatorname{succ} A) = \mathcal{G}(A, B)$,
- for all A, *psi* such that $A \in \mathcal{B}$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $psi = \mathcal{A} \restriction A$ holds $\mathcal{A}(A) = \mathcal{H}(A, psi)$.

The scheme *OS_Def* deals with a constant $\mathcal{A}$ that is an ordinal number, a constant $\mathcal{B}$ that is an ordinal number, a binary functor $\mathcal{F}$ yielding an ordinal number and a binary functor $\mathcal{G}$ yielding an ordinal number and states that:

(i) there exist A, fi such that $A = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} \mathcal{A}$ and $fi(\mathbf{0}) = \mathcal{B}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = \mathcal{F}(C, D)$ and for all C, *psi* such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{G}(C, psi)$,

(ii) for all A_1, A_2 such that there exists fi such that $A_1 = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} \mathcal{A}$ and $fi(\mathbf{0}) = \mathcal{B}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = \mathcal{F}(C, D)$ and for all C, *psi* such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{G}(C, psi)$ and there exists fi such that $A_2 = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} \mathcal{A}$ and $fi(\mathbf{0}) = \mathcal{B}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = \mathcal{F}(C, D)$ and for all C, *psi* such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{G}(C, psi)$ holds $A_1 = A_2$.

for all values of the parameters.

The scheme *OS_Result0* concerns a unary functor $\mathcal{F}$ yielding an ordinal number, a constant $\mathcal{A}$ that is an ordinal number, a binary functor $\mathcal{G}$ yielding an ordinal number and a binary functor $\mathcal{H}$ yielding an ordinal number and states that:

$\mathcal{F}(\mathbf{0}) = \mathcal{A}$

provided the parameters satisfy the following condition:

- Given A, B. Then $B = \mathcal{F}(A)$ if and only if there exists fi such that $B = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} A$ and $fi(\mathbf{0}) = \mathcal{A}$ and for all C, D such that $\mathrm{succ}\, C \in \mathrm{succ}\, A$ and $D = fi(C)$ holds $fi(\mathrm{succ}\, C) = \mathcal{G}(C, D)$ and for all C, psi such that $C \in \mathrm{succ}\, A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{H}(C, psi)$.

The scheme *OS_ResultS* deals with a constant $\mathcal{A}$ that is an ordinal number, a binary functor $\mathcal{F}$ yielding an ordinal number, a binary functor $\mathcal{G}$ yielding an ordinal number and a unary functor $\mathcal{H}$ yielding an ordinal number and states that:

for every A holds $\mathcal{H}(\mathrm{succ}\, A) = \mathcal{F}(A, \mathcal{H}(A))$

provided the parameters satisfy the following condition:

- Given A, B. Then $B = \mathcal{H}(A)$ if and only if there exists fi such that $B = \mathrm{last}\, fi$ and $\mathrm{dom}\, fi = \mathrm{succ}\, A$ and $fi(\mathbf{0}) = \mathcal{A}$ and for all C, D such that $\mathrm{succ}\, C \in \mathrm{succ}\, A$ and $D = fi(C)$ holds $fi(\mathrm{succ}\, C) = \mathcal{F}(C, D)$ and for all C, psi such that $C \in \mathrm{succ}\, A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{G}(C, psi)$.

The scheme *OS_ResultL* deals with a constant $\mathcal{A}$ that is a sequence of ordinal numbers, a constant $\mathcal{B}$ that is an ordinal number, a unary functor $\mathcal{F}$ yielding an ordinal number, a constant $\mathcal{C}$ that is an ordinal number, a binary functor $\mathcal{G}$ yielding an ordinal number and a binary functor $\mathcal{H}$ yielding an ordinal number and states that:

$\mathcal{F}(\mathcal{B}) = \mathcal{H}(\mathcal{B}, \mathcal{A})$

provided the parameters satisfy the following conditions:

- Given A, B. Then $B = \mathcal{F}(A)$ if and only if there exists fi such that $B = \mathrm{last}\, fi$ and $\mathrm{dom}\, fi = \mathrm{succ}\, A$ and $fi(\mathbf{0}) = \mathcal{C}$ and for all C, D such that $\mathrm{succ}\, C \in \mathrm{succ}\, A$ and $D = fi(C)$ holds $fi(\mathrm{succ}\, C) = \mathcal{G}(C, D)$ and for all C, psi such that $C \in \mathrm{succ}\, A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{H}(C, psi)$.
- $\mathcal{B} \neq \mathbf{0}$ and $\mathcal{B}$ is a limit ordinal number,
- $\mathrm{dom}\, \mathcal{A} = \mathcal{B}$,
- for every A such that $A \in \mathcal{B}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$.

We now define two new functors. Let us consider L. The functor $\sup L$ yields an ordinal number and is defined by:

$\sup L = \sup(\mathrm{rng}\, L)$.

The functor $\inf L$ yielding an ordinal number, is defined by:

$\inf L = \inf(\mathrm{rng}\, L)$.

One can prove the following proposition

(35) $\sup L = \sup(\mathrm{rng}\, L)$ and $\inf L = \inf(\mathrm{rng}\, L)$.

We now define two new functors. Let us consider L. The functor $\limsup L$ yielding an ordinal number, is defined by:

there exists fi such that $\limsup L = \inf fi$ and $\mathrm{dom}\, fi = \mathrm{dom}\, L$ and for every A such that $A \in \mathrm{dom}\, L$ holds $fi(A) = \sup(\mathrm{rng}(L \restriction (\mathrm{dom}\, L \setminus A)))$.

The functor $\liminf L$ yields an ordinal number and is defined by:

there exists fi such that $\liminf L = \sup fi$ and $\mathrm{dom}\, fi = \mathrm{dom}\, L$ and for every A such that $A \in \mathrm{dom}\, L$ holds $fi(A) = \inf(\mathrm{rng}(L \restriction (\mathrm{dom}\, L \setminus A)))$.

One can prove the following propositions:

(36) $A = \limsup L$ if and only if there exists fi such that $A = \inf fi$ and $\operatorname{dom} fi = \operatorname{dom} L$ and for every B such that $B \in \operatorname{dom} L$ holds $fi(B) = \sup(\operatorname{rng}(L \restriction (\operatorname{dom} L \setminus B)))$.

(37) $A = \liminf L$ if and only if there exists fi such that $A = \sup fi$ and $\operatorname{dom} fi = \operatorname{dom} L$ and for every B such that $B \in \operatorname{dom} L$ holds $fi(B) = \inf(\operatorname{rng}(L \restriction (\operatorname{dom} L \setminus B)))$.

Let us consider A, fi. The predicate A is the limit of fi is defined by:

there exists B such that $B \in \operatorname{dom} fi$ and for every C such that $B \subseteq C$ and $C \in \operatorname{dom} fi$ holds $fi(C) = \mathbf{0}$ if $A = \mathbf{0}$, for all B, C such that $B \in A$ and $A \in C$ there exists D such that $D \in \operatorname{dom} fi$ and for every ordinal number E such that $D \subseteq E$ and $E \in \operatorname{dom} fi$ holds $B \in fi(E)$ and $fi(E) \in C$, otherwise.

One can prove the following propositions:

(38) If $A = \mathbf{0}$, then A is the limit of fi if and only if there exists B such that $B \in \operatorname{dom} fi$ and for every C such that $B \subseteq C$ and $C \in \operatorname{dom} fi$ holds $fi(C) = \mathbf{0}$.

(39) If $A \neq \mathbf{0}$, then A is the limit of fi if and only if for all B, C such that $B \in A$ and $A \in C$ there exists D such that $D \in \operatorname{dom} fi$ and for every ordinal number E such that $D \subseteq E$ and $E \in \operatorname{dom} fi$ holds $B \in fi(E)$ and $fi(E) \in C$.

Let us consider fi. Let us assume that there exists A such that A is the limit of fi. The functor $\lim fi$ yielding an ordinal number, is defined by:

$\lim fi$ is the limit of fi.

Let us consider A, fi. Let us assume that $A \in \operatorname{dom} fi$. The functor $\lim_A fi$ yields an ordinal number and is defined by:

$\lim_A fi = \lim fi \restriction A$.

Next we state two propositions:

(40) If A is the limit of fi, then $\lim fi = A$.

(41) If $A \in \operatorname{dom} fi$, then $\lim_A fi = \lim fi \restriction A$.

We now define two new predicates. Let L be a sequence of ordinal numbers. The predicate L is increasing is defined by:

for all A, B such that $A \in B$ and $B \in \operatorname{dom} L$ holds $L(A) \in L(B)$.

The predicate L is continuous is defined by:

for all A, B such that $A \in \operatorname{dom} L$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $B = L(A)$ holds B is the limit of $L \restriction A$.

We now state two propositions:

(42) fi is increasing if and only if for all A, B such that $A \in B$ and $B \in \operatorname{dom} fi$ holds $fi(A) \in fi(B)$.

(43) fi is continuous if and only if for all A, B such that $A \in \operatorname{dom} fi$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $B = fi(A)$ holds B is the limit of $fi \restriction A$.

Let us consider A, B. The functor $A+B$ yielding an ordinal number, is defined by:

there exists fi such that $A+B = \mathrm{last}\, fi$ and $\mathrm{dom}\, fi = \mathrm{succ}\, B$ and $fi(\mathbf{0}) = A$ and for all C, D such that $\mathrm{succ}\, C \in \mathrm{succ}\, B$ and $D = fi(C)$ holds $fi(\mathrm{succ}\, C) = \mathrm{succ}\, D$ and for all C, psi such that $C \in \mathrm{succ}\, B$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \sup psi$.

Let us consider A, B. The functor $A \cdot B$ yielding an ordinal number, is defined by:

there exists fi such that $A \cdot B = \mathrm{last}\, fi$ and $\mathrm{dom}\, fi = \mathrm{succ}\, A$ and $fi(\mathbf{0}) = \mathbf{0}$ and for all C, D such that $\mathrm{succ}\, C \in \mathrm{succ}\, A$ and $D = fi(C)$ holds $fi(\mathrm{succ}\, C) = D + B$ and for all C, psi such that $C \in \mathrm{succ}\, A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \bigcup \sup psi$.

Let us consider A, B. The functor A^B yields an ordinal number and is defined by:

there exists fi such that $A^B = \mathrm{last}\, fi$ and $\mathrm{dom}\, fi = \mathrm{succ}\, B$ and $fi(\mathbf{0}) = \mathbf{1}$ and for all C, D such that $\mathrm{succ}\, C \in \mathrm{succ}\, B$ and $D = fi(C)$ holds $fi(\mathrm{succ}\, C) = A \cdot D$ and for all C, psi such that $C \in \mathrm{succ}\, B$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \lim psi$.

The following propositions are true:

(44) $A + \mathbf{0} = A$.

(45) $A + \mathrm{succ}\, B = \mathrm{succ}(A + B)$.

(46) If $B \neq \mathbf{0}$ and B is a limit ordinal number, then for every fi such that $\mathrm{dom}\, fi = B$ and for every C such that $C \in B$ holds $fi(C) = A + C$ holds $A + B = \sup fi$.

(47) $\mathbf{0} + A = A$.

(48) $A + \mathbf{1} = \mathrm{succ}\, A$.

(49) If $A \in B$, then $C + A \in C + B$.

(50) If $A \subseteq B$, then $C + A \subseteq C + B$.

(51) If $A \subseteq B$, then $A + C \subseteq B + C$.

(52) $\mathbf{0} \cdot A = \mathbf{0}$.

(53) $\mathrm{succ}\, B \cdot A = B \cdot A + A$.

(54) If $B \neq \mathbf{0}$ and B is a limit ordinal number, then for every fi such that $\mathrm{dom}\, fi = B$ and for every C such that $C \in B$ holds $fi(C) = C \cdot A$ holds $B \cdot A = \bigcup \sup fi$.

(55) $A \cdot \mathbf{0} = \mathbf{0}$.

(56) $\mathbf{1} \cdot A = A$ and $A \cdot \mathbf{1} = A$.

(57) If $C \neq \mathbf{0}$ and $A \in B$, then $A \cdot C \in B \cdot C$.

(58) If $A \subseteq B$, then $A \cdot C \subseteq B \cdot C$.

(59) If $A \subseteq B$, then $C \cdot A \subseteq C \cdot B$.

(60) $A^{\mathbf{0}} = \mathbf{1}$.

(61) $A^{\mathrm{succ}\, B} = A \cdot (A^B)$.

(62) If $B \neq \mathbf{0}$ and B is a limit ordinal number, then for every fi such that $\mathrm{dom}\, fi = B$ and for every C such that $C \in B$ holds $fi(C) = A^C$ holds $A^B = \lim fi$.

(63) $A^{\mathbf{1}} = A$ and $\mathbf{1}^A = \mathbf{1}$.

Let us consider A. The predicate A is natural is defined by:
$A \in \omega$.

One can prove the following propositions:

(64) A is natural if and only if $A \in \omega$.

(65) For every A there exist B, C such that B is a limit ordinal number and C is natural and $A = B + C$.

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