FORMALIZED MATHEMATICS
Vol.1, No.2, March–April 1990
Université Catholique de Louvain

Construction of a bilinear symmetric form in orthogonal vector space [1]

Eugeniusz Kusak
Warsaw University
Białystok

Wojciech Leończuk
Warsaw University
Białystok

Michał Muzalewski
Warsaw University
Białystok

Summary. In this text we present unpublished results by Eugeniusz Kusak and Wojciech Leończuk. They contain an axiomatic description of the class of all spaces $\langle V;\ \perp_\xi\rangle$, where V is a vector space over a field F, $\xi : V \times V \to F$ is a bilinear symmetric form i.e. $\xi(x,y) = \xi(y,x)$ and $x \perp_\xi y$ iff $\xi(x,y) = 0$ for $x,\ y \in V$. They also contain an effective construction of bilinear symmetric form ξ for given orthogonal space $\langle V;\ \perp\rangle$ such that $\perp=\perp_\xi$. The basic tool used in this method is the notion of orthogonal projection $\mathrm{J}(a,b,x)$ for $a,b,x \in V$. We should stress the fact that axioms of orthogonal and symplectic spaces differ only by one axiom, namely: $x \perp y+\varepsilon z \,\&\, y \perp z+\varepsilon x \Rightarrow z \perp x+\varepsilon y$. For $\varepsilon = -1$ we get the axiom on three perpendiculars characterizing orthogonal geometry. For $\varepsilon = +1$ we get the axiom characterizing symplectic geometry - see [1].

MML Identifier: ORTSP_1.

The papers [2], and [3] provide the terminology and notation for this paper. In the sequel F will be a field. We consider orthogonality structures which are systems

⟨ scalars, a carrier, an orthogonality ⟩

where the scalars is a field, the carrier is a vector space over the scalars, and the orthogonality is a relation on the carrier of the carrier of the carrier. The arguments of the notions defined below are the following: O which is an orthogonality structure; a, b which are elements of the carrier of the carrier of O. The predicate $a \perp b$ is defined by:

$\langle a,b\rangle \in$the orthogonality of O.

The following proposition is true

(1) For every O being an orthogonality structure for all elements a, b of the carrier of the carrier of O holds $a \perp b$ if and only if $\langle a,b\rangle \in$the orthogonality of O.

[1]Supported by RPBP.III-24.C6.

ISSN 0777–4028

The mode orthogonality space, which widens to the type an orthogonality structure, is defined by:

Let a, b, c, d, x be elements of the carrier of the carrier of it . Let l be an element of the carrier of the scalars of it . Then

(i) if $a \neq \Theta_{\text{the carrier of it}}$ and $b \neq \Theta_{\text{the carrier of it}}$ and $c \neq \Theta_{\text{the carrier of it}}$ and $d \neq \Theta_{\text{the carrier of it}}$, then there exists p being an element of the carrier of the carrier of it such that $p \not\perp a$ and $p \not\perp b$ and $p \not\perp c$ and $p \not\perp d$,

(ii) if $a \perp b$, then $l \cdot a \perp b$,

(iii) if $b \perp a$ and $c \perp a$, then $b + c \perp a$,

(iv) if $b \not\perp a$, then there exists k being an element of the carrier of the scalars of it such that $x - k \cdot b \perp a$,

(v) if $a \perp b - c$ and $b \perp c - a$, then $c \perp a - b$.

In the sequel S will denote an orthogonality structure. Next we state a proposition

(2) The following conditions are equivalent:

  (i) for all elements a, b, c, d, x of the carrier of the carrier of S for every element l of the carrier of the scalars of S holds if $a \neq \Theta_{\text{the carrier of } S}$ and $b \neq \Theta_{\text{the carrier of } S}$ and $c \neq \Theta_{\text{the carrier of } S}$ and $d \neq \Theta_{\text{the carrier of } S}$, then there exists p being an element of the carrier of the carrier of S such that $p \not\perp a$ and $p \not\perp b$ and $p \not\perp c$ and $p \not\perp d$ but if $a \perp b$, then $l \cdot a \perp b$ but if $b \perp a$ and $c \perp a$, then $b + c \perp a$ but if $b \not\perp a$, then there exists k being an element of the carrier of the scalars of S such that $x - k \cdot b \perp a$ but if $a \perp b - c$ and $b \perp c - a$, then $c \perp a - b$,

  (ii) S is an orthogonality space.

We adopt the following convention: S denotes an orthogonality space, a, b, c, d, p, q, x, y, z denote elements of the carrier of the carrier of S, and k, l denote elements of the carrier of the scalars of S. Let us consider S. The functor 0_S yielding an element of the carrier of the scalars of S, is defined by:

$0_S = 0_{\text{the scalars of } S}$.

One can prove the following proposition

(3) $0_S = 0_{\text{the scalars of } S}$.

Let us consider S. The functor Ω_S yields an element of the carrier of the scalars of S and is defined by:

$\Omega_S = 1_{\text{the scalars of } S}$.

The following proposition is true

(4) $\Omega_S = 1_{\text{the scalars of } S}$.

Let us consider S. The functor Θ_S yields an element of the carrier of the carrier of S and is defined by:

$\Theta_S = \Theta_{\text{the carrier of } S}$.

One can prove the following propositions:

(5) $\Theta_S = \Theta_{\text{the carrier of } S}$.

(6) If $a \neq \Theta_S$ and $b \neq \Theta_S$ and $c \neq \Theta_S$ and $d \neq \Theta_S$, then there exists p such that $p \not\perp a$ and $p \not\perp b$ and $p \not\perp c$ and $p \not\perp d$.

(7) If $a \perp b$, then $l \cdot a \perp b$.

(8) If $b \perp a$ and $c \perp a$, then $b + c \perp a$.

(9) If $b \not\perp a$, then there exists k such that $x - k \cdot b \perp a$.

(10) If $a \perp b - c$ and $b \perp c - a$, then $c \perp a - b$.

(11) $\Theta_S \perp a$.

(12) If $a \perp b$, then $b \perp a$.

(13) If $a \not\perp b$ and $c + a \perp b$, then $c \not\perp b$.

(14) If $b \not\perp a$ and $c \perp a$, then $b + c \not\perp a$.

(15) If $b \not\perp a$ and $l \neq 0_S$, then $l \cdot b \not\perp a$ and $b \not\perp l \cdot a$.

(16) If $a \perp b$, then $-a \perp b$.

(17) If $a + b \perp c$ and $a \perp c$, then $b \perp c$.

(18) If $a + b \perp c$ and $b \perp c$, then $a \perp c$.

(19) If $a - b \perp d$ and $a - c \perp d$, then $b - c \perp d$.

(20) If $b \not\perp a$ and $x - k \cdot b \perp a$ and $x - l \cdot b \perp a$, then $k = l$.

(21) If $a \perp a$ and $b \perp b$, then $a + b \perp a - b$.

(22) If $\Omega_S + \Omega_S \neq 0_S$ and there exists a such that $a \neq \Theta_S$, then there exists b such that $b \not\perp b$.

Let us consider S, a, b, x. Let us assume that $b \not\perp a$. The functor $J(a, b, x)$ yielding an element of the carrier of the scalars of S, is defined by:

for every element l of the carrier of the scalars of S such that $x - l \cdot b \perp a$ holds $J(a, b, x) = l$.

Next we state a number of propositions:

(23) If $b \not\perp a$ and $x - l \cdot b \perp a$, then $J(a, b, x) = l$.

(24) If $b \not\perp a$, then $x - J(a, b, x) \cdot b \perp a$.

(25) If $b \not\perp a$, then $J(a, b, l \cdot x) = l \cdot J(a, b, x)$.

(26) If $b \not\perp a$, then $J(a, b, x + y) = J(a, b, x) + J(a, b, y)$.

(27) If $b \not\perp a$ and $l \neq 0_S$, then $J(a, l \cdot b, x) = l^{-1} \cdot J(a, b, x)$.

(28) If $b \not\perp a$ and $l \neq 0_S$, then $J(l \cdot a, b, x) = J(a, b, x)$.

(29) If $b \not\perp a$ and $p \perp a$, then $J(a, b + p, c) = J(a, b, c)$ and $J(a, b, c + p) = J(a, b, c)$.

(30) If $b \not\perp a$ and $p \perp b$ and $p \perp c$, then $J(a + p, b, c) = J(a, b, c)$.

(31) If $b \not\perp a$ and $c - b \perp a$, then $J(a, b, c) = \Omega_S$.

(32) If $b \not\perp a$, then $J(a, b, b) = \Omega_S$.

(33) If $b \not\perp a$, then $x \perp a$ if and only if $J(a, b, x) = 0_S$.

(34) If $b \not\perp a$ and $q \not\perp a$, then $J(a, b, p) \cdot J(a, b, q)^{-1} = J(a, q, p)$.

(35) If $b \not\perp a$ and $c \not\perp a$, then $J(a, b, c) = J(a, c, b)^{-1}$.

(36) If $b \not\perp a$ and $b \perp c + a$, then $J(a, b, c) = -J(c, b, a)$.

(37) If $a \not\perp b$ and $c \not\perp b$, then $J(c, b, a) = J(b, a, c)^{-1} \cdot J(a, b, c)$.

(38) If $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$, then $J(a, q, p) \cdot J(p, a, x) = J(q, a, x) \cdot J(x, q, p)$.

(39) Suppose $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$ and $b \not\perp a$. Then $(J(a,b,p) \cdot J(p,a,x)) \cdot J(x,p,y) = (J(a,b,q) \cdot J(q,a,x)) \cdot J(x,q,y)$.

(40) If $a \not\perp p$ and $x \not\perp p$ and $y \not\perp p$, then $J(p,a,x) \cdot J(x,p,y) = J(p,a,y) \cdot J(y,p,x)$.

Let us consider S, x, y, a, b. Let us assume that $b \not\perp a$. The functor $x \cdot_{a,b} y$ yielding an element of the carrier of the scalars of S, is defined by:

for every q such that $q \not\perp a$ and $q \not\perp x$ holds $x \cdot_{a,b} y = (J(a,b,q) \cdot J(q,a,x)) \cdot J(x,q,y)$ if there exists p such that $p \not\perp a$ and $p \not\perp x$, $x \cdot_{a,b} y = 0_S$ if for every p holds $p \perp a$ or $p \perp x$.

One can prove the following propositions:

(41) If $b \not\perp a$ and $p \not\perp a$ and $p \not\perp x$, then $x \cdot_{a,b} y = (J(a,b,p) \cdot J(p,a,x)) \cdot J(x,p,y)$.

(42) If $b \not\perp a$ and for every p holds $p \perp a$ or $p \perp x$, then $x \cdot_{a,b} y = 0_S$.

(43) If $b \not\perp a$ and $x = \Theta_S$, then $x \cdot_{a,b} y = 0_S$.

(44) If $b \not\perp a$, then $x \cdot_{a,b} y = 0_S$ if and only if $y \perp x$.

(45) If $b \not\perp a$, then $x \cdot_{a,b} y = y \cdot_{a,b} x$.

(46) If $b \not\perp a$, then $x \cdot_{a,b} (l \cdot y) = l \cdot x \cdot_{a,b} y$.

(47) If $b \not\perp a$, then $x \cdot_{a,b} (y + z) = x \cdot_{a,b} y + x \cdot_{a,b} z$.

References

[1] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Construction of a bilinear antisymmetric form in symplectic vector space. *Formalized Mathematics*, 1(**2**):349–352, 1990.

[2] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[3] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received November 23, 1989