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A First Order Language

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Summary. In the paper a first order language is constructed. It includes the universal quantifier and the following propositional connectives: truth, negation, and conjunction. The variables are divided into three kinds: bound variables, fixed variables, and free variables. An infinite number of predicates for each arity is provided. Schemes of structural induction and schemes justifying definitions by structural induction have been proved. The concept of a closed formula (a formula without free occurrences of bound variables) is introduced.

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The articles [7], [8], [5], [1], [3], [4], [6], and [2] provide the notation and terminology for this paper. The following propositions are true:

(1) For all non-empty sets D_1, D_2 for every element k of D_1 holds $[\!: \{k\}, D_2 :\!] \subseteq [\!: D_1, D_2 :\!]$.

(2) For all non-empty sets D_1, D_2 for all elements k_1, k_2, k_3 of D_1 holds $[\!: \{k_1, k_2, k_3\}, D_2 :\!] \subseteq [\!: D_1, D_2 :\!]$.

In the sequel k, l denote natural numbers. The constant Var is a non-empty set and is defined by:

$\mathrm{Var} = [\!: \{4, 5, 6\}, \mathbb{N} :\!]$.

Next we state two propositions:

(3) $\mathrm{Var} = [\!: \{4, 5, 6\}, \mathbb{N} :\!]$.

(4) $\mathrm{Var} \subseteq [\!: \mathbb{N}, \mathbb{N} :\!]$.

We now define five new constructions. A variable is an element of Var.

The constant BoundVar is a non-empty subset of Var and is defined by:

$\mathrm{BoundVar} = [\!: \{4\}, \mathbb{N} :\!]$.

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The constant FixedVar is a non-empty subset of Var and is defined by:
FixedVar $= [\!: \{5\}, \mathbb{N} :\!]$.
The constant FreeVar is a non-empty subset of Var and is defined by:
FreeVar $= [\!: \{6\}, \mathbb{N} :\!]$.
The constant PredSym is a non-empty set and is defined by:
PredSym $= \{\langle k, l\rangle : 7 \leq k\}$.

The following propositions are true:

(5) For every element IT of Var holds IT is a variable.

(6) BoundVar $= [\!: \{4\}, \mathbb{N} :\!]$.

(7) FixedVar $= [\!: \{5\}, \mathbb{N} :\!]$.

(8) FreeVar $= [\!: \{6\}, \mathbb{N} :\!]$.

(9) PredSym $= \{\langle k, l\rangle : 7 \leq k\}$.

(10) PredSym $\subseteq [\!: \mathbb{N}, \mathbb{N} :\!]$.

A predicate symbol is an element of PredSym.

The following proposition is true

(11) For every element IT of PredSym holds IT is a predicate symbol.

Let P be an element of PredSym. The functor Arity(P) yielding a natural number, is defined by:
$P_{\mathbf{1}} = 7 + \text{Arity}(P)$.

Next we state a proposition

(12) For every predicate symbol P for every natural number IT holds $IT =$ Arity(P) if and only if $P_{\mathbf{1}} = 7 + IT$.

In the sequel P will denote a predicate symbol. Let us consider k. The functor PredSym$_k$ yields a non-empty subset of PredSym and is defined by:
PredSym$_k = \{P : \text{Arity}(P) = k\}$.

Next we state a proposition

(13) For every natural number k for every non-empty subset IT of PredSym holds $IT =$ PredSym$_k$ if and only if $IT = \{P : \text{Arity}(P) = k\}$.

We now define four new modes. A bound variable is an element of BoundVar.
A fixed variable is an element of FixedVar.
A free variable is an element of FreeVar.
Let us consider k. A k-ary predicate symbol is an element of PredSym$_k$.

One can prove the following four propositions:

(14) For every element IT of BoundVar holds IT is a bound variable.

(15) For every element IT of FixedVar holds IT is a fixed variable.

(16) For every element IT of FreeVar holds IT is a free variable.

(17) For every natural number k for every element IT of PredSym$_k$ holds IT is a k-ary predicate symbol.

Let k be a natural number. The mode list of variables of the length k, which widens to the type a finite sequence of elements of Var, is defined by:
len it $= k$.

One can prove the following proposition

(18) For every natural number k for every finite sequence IT of elements of Var holds IT is a list of variables of the length k if and only if $\operatorname{len} IT = k$.

Let D be a non-empty set. The predicate D is closed is defined by:
(i) D is a subset of $[:\mathbb{N}, \mathbb{N}:]^*$,
(ii) for every natural number k for every k-ary predicate symbol p for every list of variables ll of the length k holds $\langle p \rangle ^\frown ll \in D$,
(iii) $\langle \langle 0, 0 \rangle \rangle \in D$,
(iv) for every finite sequence p of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ holds $\langle \langle 1, 0 \rangle \rangle ^\frown p \in D$,
(v) for all finite sequences p, q of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ and $q \in D$ holds $(\langle \langle 2, 0 \rangle \rangle ^\frown p) ^\frown q \in D$,
(vi) for every bound variable x for every finite sequence p of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ holds $(\langle \langle 3, 0 \rangle \rangle ^\frown \langle x \rangle) ^\frown p \in D$.

We now state a proposition

(19) Let D be a non-empty set. Then D is closed if and only if the following conditions are satisfied:
  (i) D is a subset of $[:\mathbb{N}, \mathbb{N}:]^*$,
  (ii) for every natural number k for every k-ary predicate symbol p for every list of variables ll of the length k holds $\langle p \rangle ^\frown ll \in D$,
  (iii) $\langle \langle 0, 0 \rangle \rangle \in D$,
  (iv) for every finite sequence p of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ holds $\langle \langle 1, 0 \rangle \rangle ^\frown p \in D$,
  (v) for all finite sequences p, q of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ and $q \in D$ holds $(\langle \langle 2, 0 \rangle \rangle ^\frown p) ^\frown q \in D$,
  (vi) for every bound variable x for every finite sequence p of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ holds $(\langle \langle 3, 0 \rangle \rangle ^\frown \langle x \rangle) ^\frown p \in D$.

The constant WFF is a non-empty set and is defined by:

WFF is closed and for every non-empty set D such that D is closed holds $\mathrm{WFF} \subseteq D$.

Next we state two propositions:

(20) For every non-empty set IT holds $IT = \mathrm{WFF}$ if and only if IT is closed and for every non-empty set D such that D is closed holds $IT \subseteq D$.

(21) WFF is closed.

A formula is an element of WFF.

The following proposition is true

(22) For every element x of WFF holds x is a formula.

The arguments of the notions defined below are the following: P which is a predicate symbol; l which is a finite sequence of elements of Var. Let us assume that $\operatorname{Arity}(P) = \operatorname{len} l$. The functor $P ^\frown l$ yields an element of WFF and is defined by:

$P ^\frown l = \langle P \rangle ^\frown l$.

We now state a proposition

(23) For every natural number k for every k-ary predicate symbol p for every list of variables ll of the length k holds $p \frown ll = \langle p \rangle \frown ll$.

Let p be an element of WFF. The functor $@p$ yields a finite sequence of elements of $[\!: \mathbb{N}, \mathbb{N} :\!]$ and is defined by:

$@p = p$.

One can prove the following proposition

(24) For every element p of WFF holds $@p = p$.

We now define three new functors. The constant VERUM is a formula and is defined by:

$\mathrm{VERUM} = \langle \langle 0, 0 \rangle \rangle$.

Let p be an element of WFF. The functor $\neg p$ yielding a formula, is defined by:

$\neg p = \langle \langle 1, 0 \rangle \rangle \frown @p$.

Let q be an element of WFF. The functor $p \wedge q$ yields a formula and is defined by:

$p \wedge q = (\langle \langle 2, 0 \rangle \rangle \frown @p) \frown @q$.

We now state three propositions:

(25) $\mathrm{VERUM} = \langle \langle 0, 0 \rangle \rangle$.

(26) For every element p of WFF holds $\neg p = \langle \langle 1, 0 \rangle \rangle \frown @p$.

(27) For all elements p, q of WFF holds $p \wedge q = (\langle \langle 2, 0 \rangle \rangle \frown @p) \frown @q$.

The arguments of the notions defined below are the following: x which is a bound variable; p which is an element of WFF. The functor $\forall_x p$ yields a formula and is defined by:

$\forall_x p = (\langle \langle 3, 0 \rangle \rangle \frown \langle x \rangle) \frown @p$.

The following proposition is true

(28) For every bound variable x for every element p of WFF holds $\forall_x p = (\langle \langle 3, 0 \rangle \rangle \frown \langle x \rangle) \frown @p$.

The scheme *QC_Ind* deals with a unary predicate $\mathcal{P}$ and states that:

for every element F of WFF holds $\mathcal{P}[F]$

provided the parameter satisfies the following conditions:

- for every natural number k for every k-ary predicate symbol P for every list of variables ll of the length k holds $\mathcal{P}[P \frown ll]$,
- $\mathcal{P}[\mathrm{VERUM}]$,
- for every element p of WFF such that $\mathcal{P}[p]$ holds $\mathcal{P}[\neg p]$,
- for all elements p, q of WFF such that $\mathcal{P}[p]$ and $\mathcal{P}[q]$ holds $\mathcal{P}[p \wedge q]$,
- for every bound variable x for every element p of WFF such that $\mathcal{P}[p]$ holds $\mathcal{P}[\forall_x p]$.

We now define four new predicates. Let F be an element of WFF. The predicate F is atomic is defined by:

there exists k being a natural number such that there exists p being a k-ary predicate symbol such that there exists ll being a list of variables of the length k such that $F = p \frown ll$.

The predicate F is negative is defined by:

there exists p being an element of WFF such that $F = \neg p$.

The predicate F is conjunctive is defined by:

there exist p, q being elements of WFF such that $F = p \wedge q$.
The predicate F is universal is defined by:

there exists x being a bound variable such that there exists p being an element of WFF such that $F = \forall_x p$.

We now state several propositions:

(29) For every element F of WFF holds F is atomic if and only if there exists k being a natural number such that there exists p being a k-ary predicate symbol such that there exists ll being a list of variables of the length k such that $F = p \frown ll$.

(30) For every element F of WFF holds F is negative if and only if there exists p being an element of WFF such that $F = \neg p$.

(31) For every element F of WFF holds F is conjunctive if and only if there exist p, q being elements of WFF such that $F = p \wedge q$.

(32) For every element F of WFF holds F is universal if and only if there exists x being a bound variable such that there exists p being an element of WFF such that $F = \forall_x p$.

(33) For every element F of WFF holds $F =$ VERUM or F is atomic or F is negative or F is conjunctive or F is universal.

(34) For every element F of WFF holds $1 \leq \mathrm{len}(@F)$.

One can prove the following proposition

(35) For every natural number k for every k-ary predicate symbol P holds $\mathrm{Arity}(P) = k$.

In the sequel F, G are elements of WFF and s is a finite sequence. The following two propositions are true:

(36) (i) If $(@F(1))_{\mathbf{1}} = 0$, then $F =$ VERUM,

(ii) if $(@F(1))_{\mathbf{1}} = 1$, then F is negative,

(iii) if $(@F(1))_{\mathbf{1}} = 2$, then F is conjunctive,

(iv) if $(@F(1))_{\mathbf{1}} = 3$, then F is universal,

(v) if there exists k being a natural number such that $@F(1)$ is a k-ary predicate symbol, then F is atomic.

(37) If $@F = @G \frown s$, then $@F = @G$.

Let F be an element of WFF satisfying the condition: F is atomic. The functor $\mathrm{PredSym}(F)$ yielding a predicate symbol, is defined by:

there exists k being a natural number such that there exists ll being a list of variables of the length k such that there exists P being a k-ary predicate symbol such that $\mathrm{PredSym}(F) = P$ and $F = P \frown ll$.

Let F be an element of WFF satisfying the condition: F is atomic. The functor $\mathrm{Args}(F)$ yielding a finite sequence of elements of Var, is defined by:

there exists k being a natural number such that there exists P being a k-ary predicate symbol such that there exists ll being a list of variables of the length k such that $\mathrm{Args}(F) = ll$ and $F = P \frown ll$.

Next we state two propositions:

(38) For every element F of WFF such that F is atomic for every predicate symbol IT holds $IT = \mathrm{PredSym}(F)$ if and only if there exists k being a natural number such that there exists ll being a list of variables of the length k such that there exists P being a k-ary predicate symbol such that $IT = P$ and $F = P \frown ll$.

(39) For every element F of WFF such that F is atomic for every finite sequence IT of elements of Var holds $IT = \mathrm{Args}(F)$ if and only if there exists k being a natural number such that there exists P being a k-ary predicate symbol such that there exists ll being a list of variables of the length k such that $IT = ll$ and $F = P \frown ll$.

Let F be an element of WFF satisfying the condition: F is negative. The functor $\mathrm{Arg}(F)$ yields a formula and is defined by:

$F = \neg \mathrm{Arg}(F)$.

The following proposition is true

(40) For every element F of WFF such that F is negative for every formula IT holds $IT = \mathrm{Arg}(F)$ if and only if $F = \neg IT$.

Let F be an element of WFF satisfying the condition: F is conjunctive. The functor $\mathrm{LeftArg}(F)$ yielding a formula, is defined by:

there exists q being an element of WFF such that $F = \mathrm{LeftArg}(F) \wedge q$.

Let F be an element of WFF satisfying the condition: F is conjunctive. The functor $\mathrm{RightArg}(F)$ yields a formula and is defined by:

there exists p being an element of WFF such that $F = p \wedge \mathrm{RightArg}(F)$.

Next we state two propositions:

(41) For every element F of WFF such that F is conjunctive for every formula IT holds $IT = \mathrm{LeftArg}(F)$ if and only if there exists q being an element of WFF such that $F = IT \wedge q$.

(42) For every element F of WFF such that F is conjunctive for every formula IT holds $IT = \mathrm{RightArg}(F)$ if and only if there exists p being an element of WFF such that $F = p \wedge IT$.

We now define two new functors. Let F be an element of WFF satisfying the condition: F is universal. The functor $\mathrm{Bound}(F)$ yields a bound variable and is defined by:

there exists p being an element of WFF such that $F = \forall_{\mathrm{Bound}(F)} p$.

The functor $\mathrm{Scope}(F)$ yielding a formula, is defined by:

there exists x being a bound variable such that $F = \forall_x \mathrm{Scope}(F)$.

One can prove the following propositions:

(43) For every element F of WFF such that F is universal for every bound variable IT holds $IT = \mathrm{Bound}(F)$ if and only if there exists p being an element of WFF such that $F = \forall_{IT} p$.

(44) For every element F of WFF such that F is universal for every formula IT holds $IT = \mathrm{Scope}(F)$ if and only if there exists x being a bound variable such that $F = \forall_x IT$.

In the sequel p will be an element of WFF. We now state three propositions:

(45) If p is negative, then $\mathrm{len}(@\,\mathrm{Arg}(p)) < \mathrm{len}(@p)$.

(46) If p is conjunctive, then $\mathrm{len}(@\,\mathrm{LeftArg}(p)) < \mathrm{len}(@p)$ and $\mathrm{len}(@\,\mathrm{RightArg}(p)) < \mathrm{len}(@p)$.

(47) If p is universal, then $\mathrm{len}(@\,\mathrm{Scope}(p)) < \mathrm{len}(@p)$.

The scheme *QC_Ind2* concerns a unary predicate $\mathcal{P}$ and states that:

for every element p of WFF holds $\mathcal{P}[p]$

provided the parameter satisfies the following condition:

- for every element p of WFF holds if p is atomic, then $\mathcal{P}[p]$ but $\mathcal{P}[\mathrm{VERUM}]$ but if p is negative and $\mathcal{P}[\mathrm{Arg}(p)]$, then $\mathcal{P}[p]$ but if p is conjunctive and $\mathcal{P}[\mathrm{LeftArg}(p)]$ and $\mathcal{P}[\mathrm{RightArg}(p)]$, then $\mathcal{P}[p]$ but if p is universal and $\mathcal{P}[\mathrm{Scope}(p)]$, then $\mathcal{P}[p]$.

In the sequel F will denote an element of WFF. The following propositions are true:

(48) For every natural number k for every k-ary predicate symbol P holds $P_{\mathbf{1}} \neq 0$ and $P_{\mathbf{1}} \neq 1$ and $P_{\mathbf{1}} \neq 2$ and $P_{\mathbf{1}} \neq 3$.

(49) (i) $(@\,\mathrm{VERUM}(1))_{\mathbf{1}} = 0$,

(ii) if F is atomic, then there exists k being a natural number such that $@F(1)$ is a k-ary predicate symbol,

(iii) if F is negative, then $(@F(1))_{\mathbf{1}} = 1$,

(iv) if F is conjunctive, then $(@F(1))_{\mathbf{1}} = 2$,

(v) if F is universal, then $(@F(1))_{\mathbf{1}} = 3$.

(50) If F is atomic, then $(@F(1))_{\mathbf{1}} \neq 0$ and $(@F(1))_{\mathbf{1}} \neq 1$ and $(@F(1))_{\mathbf{1}} \neq 2$ and $(@F(1))_{\mathbf{1}} \neq 3$.

In the sequel p denotes an element of WFF. The following proposition is true

(51) (i) Neither VERUM is atomic nor VERUM is negative nor VERUM is conjunctive nor VERUM is universal,

(ii) for no p holds p is atomic and p is negative or p is atomic and p is conjunctive or p is atomic and p is universal or p is negative and p is conjunctive or p is negative and p is universal or p is conjunctive and p is universal.

The scheme *QC_Func_Ex* concerns a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is an element of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$ and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

there exists F being a function from WFF into $\mathcal{A}$ such that for every element p of WFF for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$.

for all values of the parameters.

In the sequel k denotes a natural number. Let ll be a finite sequence of elements of Var. The functor $\mathrm{snb}(ll)$ yields an element of 2^{BoundVar} **qua** a non-empty set and is defined by:

$\mathrm{snb}(ll) = \{ll(k) : 1 \leq k \wedge k \leq \mathrm{len}\, ll \wedge ll(k) \in \mathrm{BoundVar}\}$.

The following proposition is true

(52) For every finite sequence ll of elements of Var holds $\mathrm{snb}(ll) = \{ll(k) : 1 \leq k \wedge k \leq \mathrm{len}\, ll \wedge ll(k) \in \mathrm{BoundVar}\}$.

Let x be an element of 2^{BoundVar} **qua** a non-empty set. The functor $@x$ yields an element of 2^{BoundVar} and is defined by:

$@x = x$.

Next we state a proposition

(53) For every element x of 2^{BoundVar} **qua** a non-empty set holds $@x = x$.

Let x be an element of 2^{BoundVar}. The functor $@x$ yields an element of 2^{BoundVar} **qua** a non-empty set and is defined by:

$@x = x$.

One can prove the following proposition

(54) For every element x of 2^{BoundVar} holds $@x = x$.

Let b be a bound variable. Then $\{b\}$ is an element of 2^{BoundVar}.

Let X, Y be elements of 2^{BoundVar}. Then $X \cup Y$ is an element of 2^{BoundVar}. Then $X \setminus Y$ is an element of 2^{BoundVar}.

In the sequel k denotes a natural number. Let p be a formula. The functor $\mathrm{snb}(p)$ yields an element of 2^{BoundVar} and is defined by:

there exists F being a function from

WFF

into 2^{BoundVar} such that $\mathrm{snb}(p) = F(p)$ and for every element p of WFF holds $F(\mathrm{VERUM}) = \emptyset$ but if p is atomic, then $F(p) = \{\mathrm{Args}(p)(k) : 1 \leq k \wedge k \leq \mathrm{len}\,\mathrm{Args}(p) \wedge \mathrm{Args}(p)(k) \in \mathrm{BoundVar}\}$ but if p is negative, then $F(p) = F(\mathrm{Arg}(p))$ but if p is conjunctive, then $F(p) = @(F(\mathrm{LeftArg}(p))) \cup @(F(\mathrm{RightArg}(p)))$ but if p is universal, then $F(p) = @(F(\mathrm{Scope}(p))) \setminus \{\mathrm{Bound}(p)\}$.

We now state a proposition

(55) Let p be a formula. Let IT be an element of 2^{BoundVar}. Then $IT = \mathrm{snb}(p)$ if and only if there exists F being a function from WFF into 2^{BoundVar} such that $IT = F(p)$ and for every element p of WFF holds $F(\mathrm{VERUM}) = \emptyset$ but if p is atomic, then $F(p) = \{\mathrm{Args}(p)(k) : 1 \leq k \wedge k \leq \mathrm{len}\,\mathrm{Args}(p) \wedge \mathrm{Args}(p)(k) \in \mathrm{BoundVar}\}$ but if p is negative, then $F(p) = F(\mathrm{Arg}(p))$ but if p is conjunctive, then $F(p) = @(F(\mathrm{LeftArg}(p))) \cup @(F(\mathrm{RightArg}(p)))$ but if p is universal, then $F(p) = @(F(\mathrm{Scope}(p))) \setminus \{\mathrm{Bound}(p)\}$.

Let p be a formula. The predicate p is closed is defined by:

$\mathrm{snb}(p) = \emptyset$.

One can prove the following proposition

(56) For every formula p holds p is closed if and only if $\mathrm{snb}(p) = \emptyset$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[6] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[8] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

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