FORMALIZED MATHEMATICS
Vol.1, No.2, March–April 1990
Université Catholique de Louvain

Subspaces and Cosets of Subspaces in Real Linear Space

Wojciech A. Trybulec[1]
Warsaw University

Summary. The following notions are introduced in the article: subspace of a real linear space, zero subspace and improper subspace, coset of a subspace. The relation of a subset of the vectors being linearly closed is also introduced. Basic theorems concerning those notions are proved in the article.

MML Identifier: `RLSUB_1`.

The papers [4], [2], [6], [3], [1], and [5] provide the terminology and notation for this paper. For simplicity we follow a convention: V, X, Y are real linear spaces, u, v, v_1, v_2 are vectors of V, a is a real number, V_1, V_2, V_3 are subsets of the vectors of V, and x be arbitrary. Let us consider V, V_1. The predicate V_1 is linearly closed is defined by:

for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

Next we state a number of propositions:

(1) If for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$, then V_1 is linearly closed.

(2) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$.

(3) If V_1 is linearly closed, then for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

(4) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then $0_V \in V_1$.

(5) If V_1 is linearly closed, then for every v such that $v \in V_1$ holds $-v \in V_1$.

(6) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v - u \in V_1$.

(7) $\{0_V\}$ is linearly closed.

(8) If the vectors of $V = V_1$, then V_1 is linearly closed.

[1]Supported by RPBP.III-24.C1

ISSN 0777–4028

(9) If V_1 is linearly closed and V_2 is linearly closed and $V_3 = \{v + u : v \in V_1 \wedge u \in V_2\}$, then V_3 is linearly closed.

(10) If V_1 is linearly closed and V_2 is linearly closed, then $V_1 \cap V_2$ is linearly closed.

Let us consider V. The mode subspace of V, which widens to the type a real linear space, is defined by:

the vectors of it $\subseteq$the vectors of V and the zero of it $=$the zero of V and the addition of it $=$(the addition of V) $\restriction$ [: the vectors of it, the vectors of it :] and the multiplication of it $=$(the multiplication of V) $\restriction$ [: $\mathbb{R}$, the vectors of it :].

Next we state a proposition

(11) If the vectors of X $\subseteq$the vectors of V and the zero of X $=$the zero of V and the addition of X $=$(the addition of V) $\restriction$ [: the vectors of X, the vectors of X :] and the multiplication of X $=$(the multiplication of V) $\restriction$ [: $\mathbb{R}$, the vectors of X :], then X is a subspace of V.

We follow a convention: W, W_1, W_2 will denote subspaces of V and w, w_1, w_2 will denote vectors of W. We now state a number of propositions:

(12) the vectors of W $\subseteq$the vectors of V.

(13) the zero of W $=$the zero of V.

(14) the addition of W $=$(the addition of V) $\restriction$ [: the vectors of W, the vectors of W :].

(15) the multiplication of W $=$(the multiplication of V) $\restriction$ [: $\mathbb{R}$, the vectors of W :].

(16) If $x \in W_1$ and W_1 is a subspace of W_2, then $x \in W_2$.

(17) If $x \in W$, then $x \in V$.

(18) w is a vector of V.

(19) $0_W = 0_V$.

(20) $0_{W_1} = 0_{W_2}$.

(21) If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(22) If $w = v$, then $a \cdot w = a \cdot v$.

(23) If $w = v$, then $-v = -w$.

(24) If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(25) $0_V \in W$.

(26) $0_{W_1} \in W_2$.

(27) $0_W \in V$.

(28) If $u \in W$ and $v \in W$, then $u + v \in W$.

(29) If $v \in W$, then $a \cdot v \in W$.

(30) If $v \in W$, then $-v \in W$.

(31) If $u \in W$ and $v \in W$, then $u - v \in W$.

In the sequel D is a non-empty set, d_1 is an element of D, A is a binary operation on D, and M is a function from [: $\mathbb{R}$, D :] into D. We now state a number of propositions:

(32) If $V_1 = D$ and $d_1 = 0_V$ and $A =$(the addition of V) $\restriction [: V_1, V_1 :]$ and $M =$(the multiplication of V) $\restriction [: \mathbb{R}, V_1 :]$, then $\langle D, d_1, A, M \rangle$ is a subspace of V.

(33) V is a subspace of V.

(34) If V is a subspace of X and X is a subspace of V, then $V = X$.

(35) If V is a subspace of X and X is a subspace of Y, then V is a subspace of Y.

(36) If the vectors of $W_1 \subseteq$the vectors of W_2, then W_1 is a subspace of W_2.

(37) If for every v such that $v \in W_1$ holds $v \in W_2$, then W_1 is a subspace of W_2.

(38) If the vectors of $W_1 =$the vectors of W_2, then $W_1 = W_2$.

(39) If for every v holds $v \in W_1$ if and only if $v \in W_2$, then $W_1 = W_2$.

(40) If the vectors of $W =$the vectors of V, then $W = V$.

(41) If for every v holds $v \in W$ if and only if $v \in V$, then $W = V$.

(42) If the vectors of $W = V_1$, then V_1 is linearly closed.

(43) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then there exists W such that $V_1 =$the vectors of W.

Let us consider V. The functor $\mathbf{0}_V$ yielding a subspace of V, is defined by:
the vectors of $\mathbf{0}_V = \{0_V\}$.

Let us consider V. The functor Ω_V yielding a subspace of V, is defined by:
$\Omega_V = V$.

We now state a number of propositions:

(44) the vectors of $\mathbf{0}_V = \{0_V\}$.

(45) If the vectors of $W = \{0_V\}$, then $W = \mathbf{0}_V$.

(46) $\Omega_V = V$.

(47) $\Omega_V = \mathbf{0}_V$ if and only if $V = \mathbf{0}_V$.

(48) $\mathbf{0}_W = \mathbf{0}_V$.

(49) $\mathbf{0}_{W_1} = \mathbf{0}_{W_2}$.

(50) $\mathbf{0}_W$ is a subspace of V.

(51) $\mathbf{0}_V$ is a subspace of W.

(52) $\mathbf{0}_{W_1}$ is a subspace of W_2.

(53) W is a subspace of Ω_V.

(54) V is a subspace of Ω_V.

Let us consider V, v, W. The functor $v + W$ yielding a subset of the vectors of V, is defined by:
$v + W = \{v + u : u \in W\}$.

Let us consider V, W. The mode coset of W, which widens to the type a subset of the vectors of V, is defined by:
there exists v such that it $= v + W$.

In the sequel B, C will be cosets of W. We now state a number of propositions:

(55) $v + W = \{v + u : u \in W\}$.

(56) There exists v such that $C = v + W$.
(57) If $V_1 = v + W$, then V_1 is a coset of W.
(58) $0_V \in v + W$ if and only if $v \in W$.
(59) $v \in v + W$.
(60) $0_V + W =$the vectors of W.
(61) $v + \mathbf{0}_V = \{v\}$.
(62) $v + \Omega_V =$the vectors of V.
(63) $0_V \in v + W$ if and only if $v + W =$the vectors of W.
(64) $v \in W$ if and only if $v + W =$the vectors of W.
(65) If $v \in W$, then $a \cdot v + W =$the vectors of W.
(66) If $a \neq 0$ and $a \cdot v + W =$the vectors of W, then $v \in W$.
(67) $v \in W$ if and only if $(-v) + W =$the vectors of W.
(68) $u \in W$ if and only if $v + W = (v + u) + W$.
(69) $u \in W$ if and only if $v + W = (v - u) + W$.
(70) $v \in u + W$ if and only if $u + W = v + W$.
(71) $v + W = (-v) + W$ if and only if $v \in W$.
(72) If $u \in v_1 + W$ and $u \in v_2 + W$, then $v_1 + W = v_2 + W$.
(73) If $u \in v + W$ and $u \in (-v) + W$, then $v \in W$.
(74) If $a \neq 1$ and $a \cdot v \in v + W$, then $v \in W$.
(75) If $v \in W$, then $a \cdot v \in v + W$.
(76) $-v \in v + W$ if and only if $v \in W$.
(77) $u + v \in v + W$ if and only if $u \in W$.
(78) $v - u \in v + W$ if and only if $u \in W$.
(79) $u \in v + W$ if and only if there exists v_1 such that $v_1 \in W$ and $u = v + v_1$.
(80) $u \in v + W$ if and only if there exists v_1 such that $v_1 \in W$ and $u = v - v_1$.
(81) There exists v such that $v_1 \in v + W$ and $v_2 \in v + W$ if and only if $v_1 - v_2 \in W$.
(82) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v + v_1 = u$.
(83) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v - v_1 = u$.
(84) $v + W_1 = v + W_2$ if and only if $W_1 = W_2$.
(85) If $v + W_1 = u + W_2$, then $W_1 = W_2$.

In the sequel C_1 denotes a coset of W_1 and C_2 denotes a coset of W_2. We now state a number of propositions:

(86) C is linearly closed if and only if $C =$the vectors of W.
(87) If $C_1 = C_2$, then $W_1 = W_2$.
(88) $\{v\}$ is a coset of $\mathbf{0}_V$.
(89) If V_1 is a coset of $\mathbf{0}_V$, then there exists v such that $V_1 = \{v\}$.
(90) the vectors of W is a coset of W.
(91) the vectors of V is a coset of Ω_V.
(92) If V_1 is a coset of Ω_V, then $V_1 =$the vectors of V.

(93) $0_V \in C$ if and only if $C =$the vectors of W.

(94) $u \in C$ if and only if $C = u + W$.

(95) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u+v_1 = v$.

(96) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u-v_1 = v$.

(97) There exists C such that $v_1 \in C$ and $v_2 \in C$ if and only if $v_1 - v_2 \in W$.

(98) If $u \in B$ and $u \in C$, then $B = C$.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[4] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[5] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[6] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received July 24, 1989