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Vectors in Real Linear Space

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Summary. In this article we introduce a notion of real linear space, operations on vectors: addition, multiplication by real number, inverse vector, substraction. The sum of finite sequence of the vectors is also defined. Theorems that belong rather to [1] or [2] are proved.

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The notation and terminology used here have been introduced in the following articles: [7], [4], [5], [3], [6], [2], and [1]. We consider RLS structures which are systems

$\langle$ vectors, a zero, an addition, a multiplication $\rangle$

where the vectors is a non-empty set, the zero is an element of the vectors, the addition is a binary operation on the vectors, and the multiplication is a function from $[\!:\mathbb{R}, \text{the vectors}:\!]$ into the vectors. In the sequel V will denote an RLS structure, v will denote an element of the vectors of V, and x will be arbitrary. Let us consider V. A vector of V is an element of the vectors of V.

Next we state a proposition

(1) v is a vector of V.

Let us consider V, x. The predicate $x \in V$ is defined by:
$x \in$the vectors of V.

Next we state two propositions:

(2) $x \in V$ if and only if $x \in$the vectors of V.

(3) $v \in V$.

Let us consider V. The functor 0_V yielding a vector of V, is defined by:
$0_V =$the zero of V.

In the sequel v, w will denote vectors of V and a, b will denote real numbers. Let us consider V, v, w. The functor $v + w$ yields a vector of V and is defined by:

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$v + w =$(the addition of V)($\langle v, w\rangle$).

Let us consider V, v, a. The functor $a \cdot v$ yielding a vector of V, is defined by: $a \cdot v =$(the multiplication of V)($\langle a, v\rangle$).

We now state three propositions:

(4) $0_V =$the zero of V.

(5) $v + w =$(the addition of V)($\langle v, w\rangle$).

(6) $a \cdot v =$(the multiplication of V)($\langle a, v\rangle$).

The mode real linear space, which widens to the type an RLS structure, is defined by:

(i) for all vectors v, w of it holds $v + w = w + v$,
(ii) for all vectors u, v, w of it holds $(u + v) + w = u + (v + w)$,
(iii) for every vector v of it holds $v + 0_{\rm it} = v$,
(iv) for every vector v of it there exists w being a vector of it such that $v+w = 0_{\rm it}$,
(v) for every a for all vectors v, w of it holds $a \cdot (v + w) = a \cdot v + a \cdot w$,
(vi) for all a, b for every vector v of it holds $(a + b) \cdot v = a \cdot v + b \cdot v$,
(vii) for all a, b for every vector v of it holds $(a \cdot b) \cdot v = a \cdot (b \cdot v)$,
(viii) for every vector v of it holds $1 \cdot v = v$.

Next we state a proposition

(7) Suppose that
(i) for all vectors v, w of V holds $v + w = w + v$,
(ii) for all vectors u, v, w of V holds $(u + v) + w = u + (v + w)$,
(iii) for every vector v of V holds $v + 0_V = v$,
(iv) for every vector v of V there exists w being a vector of V such that $v + w = 0_V$,
(v) for every a for all vectors v, w of V holds $a \cdot (v + w) = a \cdot v + a \cdot w$,
(vi) for all a, b for every vector v of V holds $(a + b) \cdot v = a \cdot v + b \cdot v$,
(vii) for all a, b for every vector v of V holds $(a \cdot b) \cdot v = a \cdot (b \cdot v)$,
(viii) for every vector v of V holds $1 \cdot v = v$.
Then V is a real linear space.

We follow the rules: V denotes a real linear space and u, v, v_1, v_2, w denote vectors of V. The following propositions are true:

(8) $v + w = w + v$.

(9) $(u + v) + w = u + (v + w)$.

(10) $v + 0_V = v$ and $0_V + v = v$.

(11) There exists w such that $v + w = 0_V$.

(12) $a \cdot (v + w) = a \cdot v + a \cdot w$.

(13) $(a + b) \cdot v = a \cdot v + b \cdot v$.

(14) $(a \cdot b) \cdot v = a \cdot (b \cdot v)$.

(15) $1 \cdot v = v$.

Let us consider V, v. The functor $-v$ yields a vector of V and is defined by: $v + (-v) = 0_V$.

Let us consider V, v, w. The functor $v - w$ yields a vector of V and is defined by:

$v - w = v + (-w)$.

Next we state a number of propositions:

(16) $v + (-v) = 0_V$.
(17) If $v + w = 0_V$, then $w = -v$.
(18) $v - w = v + (-w)$.
(19) If $v + w = 0_V$, then $v = -w$.
(20) There exists w such that $v + w = u$.
(21) If $w + v_1 = u$ and $w + v_2 = u$, then $v_1 = v_2$.
(22) If $v + w = v$, then $w = 0_V$.
(23) If $a = 0$ or $v = 0_V$, then $a \cdot v = 0_V$.
(24) If $a \cdot v = 0_V$, then $a = 0$ or $v = 0_V$.
(25) $-0_V = 0_V$.
(26) $v - 0_V = v$.
(27) $0_V - v = -v$.
(28) $v - v = 0_V$.
(29) $-v = (-1) \cdot v$.
(30) $-(-v) = v$.
(31) If $-v = -w$, then $v = w$.
(32) If $v = -w$, then $-v = w$.
(33) If $v = -v$, then $v = 0_V$.
(34) If $v + v = 0_V$, then $v = 0_V$.
(35) If $v - w = 0_V$, then $v = w$.
(36) There exists w such that $v - w = u$.
(37) If $w - v_1 = u$ and $w - v_2 = u$, then $v_1 = v_2$.
(38) $a \cdot (-v) = (-a) \cdot v$.
(39) $a \cdot (-v) = -a \cdot v$.
(40) $(-a) \cdot (-v) = a \cdot v$.
(41) $v - (u + w) = (v - u) - w$.
(42) $(v + u) - w = v + (u - w)$.
(43) $v - (u - w) = (v - u) + w$.
(44) $-(v + w) = (-v) - w$.
(45) $-(v + w) = (-v) + (-w)$.
(46) $(-v) - w = (-w) - v$.
(47) $-(v - w) = (-v) + w$.
(48) $a \cdot (v - w) = a \cdot v - a \cdot w$.
(49) $(a - b) \cdot v = a \cdot v - b \cdot v$.
(50) If $a \neq 0$ and $a \cdot v = a \cdot w$, then $v = w$.
(51) If $v \neq 0_V$ and $a \cdot v = b \cdot v$, then $a = b$.

For simplicity we adopt the following convention: F, G denote finite sequences of elements of the vectors of V, f denotes a function from $\mathbb{N}$ into the vectors of V, j, k, n denote natural numbers, and p, q denote finite sequences. Let us consider V, f, j. Then $f(j)$ is a vector of V.

Let us consider V, v, u. Then $\langle v, u\rangle$ is a finite sequence of elements of the vectors of V.

Let us consider V, v, u, w. Then $\langle v, u, w\rangle$ is a finite sequence of elements of the vectors of V.

Let us consider V, F. The functor $\sum F$ yields a vector of V and is defined by:

there exists f such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0_V$ and for all j, v such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$.

The following propositions are true:

(52) If there exists f such that $u = f(\operatorname{len} F)$ and $f(0) = 0_V$ and for all j, v such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$, then $u = \sum F$.

(53) There exists f such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0_V$ and for all j, v such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$.

(54) If $k \in \operatorname{Seg} n$ and $\operatorname{len} F = n$, then $F(k)$ is a vector of V.

(55) If $\operatorname{len} F = \operatorname{len} G + 1$ and $G = F \restriction \operatorname{Seg}(\operatorname{len} G)$ and $v = F(\operatorname{len} F)$, then $\sum F = \sum G + v$.

(56) If $\operatorname{len} F = \operatorname{len} G$ and for all k, v such that $k \in \operatorname{Seg}(\operatorname{len} F)$ and $v = G(k)$ holds $F(k) = a \cdot v$, then $\sum F = a \cdot \sum G$.

(57) If $\operatorname{len} F = \operatorname{len} G$ and for all k, v such that $k \in \operatorname{Seg}(\operatorname{len} F)$ and $v = G(k)$ holds $F(k) = -v$, then $\sum F = -\sum G$.

(58) $\sum(F \frown G) = \sum F + \sum G$.

(59) If $\operatorname{rng} F = \operatorname{rng} G$ and F is one-to-one and G is one-to-one, then $\sum F = \sum G$.

(60) $\sum \varepsilon_{(\text{the vectors of } V)} = 0_V$.

(61) $\sum\langle v\rangle = v$.

(62) $\sum\langle v, u\rangle = v + u$.

(63) $\sum\langle v, u, w\rangle = (v + u) + w$.

(64) $a \cdot \sum \varepsilon_{(\text{the vectors of } V)} = 0_V$.

(65) $a \cdot \sum\langle v\rangle = a \cdot v$.

(66) $a \cdot \sum\langle v, u\rangle = a \cdot v + a \cdot u$.

(67) $a \cdot \sum\langle v, u, w\rangle = (a \cdot v + a \cdot u) + a \cdot w$.

(68) $-\sum \varepsilon_{(\text{the vectors of } V)} = 0_V$.

(69) $-\sum\langle v\rangle = -v$.

(70) $-\sum\langle v, u\rangle = (-v) - u$.

(71) $-\sum\langle v, u, w\rangle = ((-v) - u) - w$.

(72) $\sum\langle v, w\rangle = \sum\langle w, v\rangle$.

(73) $\sum\langle v, w\rangle = \sum\langle v\rangle + \sum\langle w\rangle$.

(74) $\sum\langle 0_V, 0_V\rangle = 0_V$.

(75) $\sum\langle 0_V, v\rangle = v$ and $\sum\langle v, 0_V\rangle = v$.

(76) $\sum\langle v, -v\rangle = 0_V$ and $\sum\langle -v, v\rangle = 0_V$.

(77) $\sum\langle v, -w\rangle = v - w$ and $\sum\langle -w, v\rangle = v - w$.

(78) $\sum\langle -v, -w\rangle = -(v + w)$ and $\sum\langle -w, -v\rangle = -(v + w)$.

(79) $\sum\langle v, v\rangle = 2 \cdot v$.

(80) $\sum\langle -v, -v\rangle = (-2) \cdot v$.

(81) $\sum\langle u, v, w\rangle = (\sum\langle u\rangle + \sum\langle v\rangle) + \sum\langle w\rangle$.

(82) $\sum\langle u, v, w\rangle = \sum\langle u, v\rangle + w$.

(83) $\sum\langle u, v, w\rangle = \sum\langle v, w\rangle + u$.

(84) $\sum\langle u, v, w\rangle = \sum\langle u, w\rangle + v$.

(85) $\sum\langle u, v, w\rangle = \sum\langle u, w, v\rangle$.

(86) $\sum\langle u, v, w\rangle = \sum\langle v, u, w\rangle$.

(87) $\sum\langle u, v, w\rangle = \sum\langle v, w, u\rangle$.

(88) $\sum\langle u, v, w\rangle = \sum\langle w, u, v\rangle$.

(89) $\sum\langle u, v, w\rangle = \sum\langle w, v, u\rangle$.

(90) $\sum\langle 0_V, 0_V, 0_V\rangle = 0_V$.

(91) $\sum\langle 0_V, 0_V, v\rangle = v$ and $\sum\langle 0_V, v, 0_V\rangle = v$ and $\sum\langle v, 0_V, 0_V\rangle = v$.

(92) $\sum\langle 0_V, u, v\rangle = u + v$ and $\sum\langle u, v, 0_V\rangle = u + v$ and $\sum\langle u, 0_V, v\rangle = u + v$.

(93) $\sum\langle v, v, v\rangle = 3 \cdot v$.

(94) If $\operatorname{len} F = 0$, then $\sum F = 0_V$.

(95) If $\operatorname{len} F = 1$, then $\sum F = F(1)$.

(96) If $\operatorname{len} F = 2$ and $v_1 = F(1)$ and $v_2 = F(2)$, then $\sum F = v_1 + v_2$.

(97) If $\operatorname{len} F = 3$ and $v_1 = F(1)$ and $v_2 = F(2)$ and $v = F(3)$, then $\sum F = (v_1 + v_2) + v$.

(98) If $j < 1$, then $j = 0$.

(99) $1 \leq k$ if and only if $k \neq 0$.

(100) $k \leq k + n$ and $k \leq n + k$.

(101) $k < k + 1$ and $k < 1 + k$.

(102) If $k \neq 0$, then $n < n + k$ and $n < k + n$.

(103) $k < k + n$ if and only if $1 \leq n$.

(104) $\operatorname{Seg} k = \operatorname{Seg}(k + 1) \setminus \{k + 1\}$.

(105) $p = (p \frown q) \restriction \operatorname{Seg}(\operatorname{len} p)$.

(106) If $\operatorname{rng} p = \operatorname{rng} q$ and p is one-to-one and q is one-to-one, then $\operatorname{len} p = \operatorname{len} q$.

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