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Construction of a bilinear antisymmetric form in symplectic vector space [1]

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Summary. In this text we will present unpublished results by Eugeniusz Kusak. It contains an axiomatic description of the class of all spaces $\langle V; \perp_\xi \rangle$, where V is a vector space over a field F, $\xi : V \times V \to F$ is a bilinear antisymmetric form i.e. $\xi(x, y) = -\xi(y, x)$ and $x \perp_\xi y$ iff $\xi(x, y) = 0$ for $x,\ y \in V$. It also contains an effective construction of bilinear antisymmetric form ξ for given symplectic space $\langle V; \perp \rangle$ such that $\perp = \perp_\xi$. The basic tool used in this method is the notion of orthogonal projection $\mathrm{J}(a, b, x)$ for $a, b, x \in V$. We should stress the fact that axioms of orthogonal and symplectic spaces differ only by one axiom, namely: $x \perp y + \varepsilon z\ \&\ y \perp z + \varepsilon x \Rightarrow z \perp x + \varepsilon y$. For $\varepsilon = +1$ we get the axiom characterizing symplectic geometry. For $\varepsilon = -1$ we get the axiom on three perpendiculars characterizing orthogonal geometry - see [1].

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The terminology and notation used in this paper have been introduced in the following papers: [2], and [3]. In the sequel F will be a field. We consider symplectic structures which are systems

⟨ scalars, a carrier, an orthogonality ⟩

where the scalars is a field, the carrier is a vector space over the scalars, and the orthogonality is a relation on the carrier of the carrier of the carrier. The arguments of the notions defined below are the following: S which is a symplectic structure; a, b which are elements of the carrier of the carrier of S. The predicate $a \perp b$ is defined by:

$\langle a, b \rangle \in$the orthogonality of S.

One can prove the following proposition

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(1) For every S being a symplectic structure for all elements a, b of the carrier of the carrier of S holds $a \perp b$ if and only if $\langle a, b \rangle \in$the orthogonality of S.

The mode symplectic space, which widens to the type a symplectic structure, is defined by:

Let a, b, c, x be elements of the carrier of the carrier of it . Let l be an element of the carrier of the scalars of it . Then

(i) if $a \neq \Theta_{\text{the carrier of it}}$, then there exists y being an element of the carrier of the carrier of it such that $y \not\perp a$,

(ii) if $a \perp b$, then $l \cdot a \perp b$,

(iii) if $b \perp a$ and $c \perp a$, then $b + c \perp a$,

(iv) if $b \not\perp a$, then there exists k being an element of the carrier of the scalars of it such that $x - k \cdot b \perp a$,

(v) if $a \perp b + c$ and $b \perp c + a$, then $c \perp a + b$.

In the sequel S is a symplectic structure. We now state a proposition

(2) The following conditions are equivalent:

(i) for all elements a, b, c, x of the carrier of the carrier of S for every element l of the carrier of the scalars of S holds if $a \neq \Theta_{\text{the carrier of } S}$, then there exists y being an element of the carrier of the carrier of S such that $y \not\perp a$ but if $a \perp b$, then $l \cdot a \perp b$ but if $b \perp a$ and $c \perp a$, then $b + c \perp a$ but if $b \not\perp a$, then there exists k being an element of the carrier of the scalars of S such that $x - k \cdot b \perp a$ but if $a \perp b + c$ and $b \perp c + a$, then $c \perp a + b$,

(ii) S is a symplectic space.

We follow the rules: S is a symplectic space, a, b, c, d, a', b', p, q, x, y, z are elements of the carrier of the carrier of S, and k, l are elements of the carrier of the scalars of S. Let us consider S. The functor 0_S yields an element of the carrier of the scalars of S and is defined by:

$0_S = 0_{\text{the scalars of } S}$.

Next we state a proposition

(3) $0_S = 0_{\text{the scalars of } S}$.

Let us consider S. The functor Ω_S yielding an element of the carrier of the scalars of S, is defined by:

$\Omega_S = 1_{\text{the scalars of } S}$.

The following proposition is true

(4) $\Omega_S = 1_{\text{the scalars of } S}$.

Let us consider S. The functor Θ_S yields an element of the carrier of the carrier of S and is defined by:

$\Theta_S = \Theta_{\text{the carrier of } S}$.

The following propositions are true:

(5) $\Theta_S = \Theta_{\text{the carrier of } S}$.

(6) If $a \neq \Theta_S$, then there exists b such that $b \not\perp a$.

(7) If $a \perp b$, then $l \cdot a \perp b$.

(8) If $b \perp a$ and $c \perp a$, then $b + c \perp a$.

(9) If $b \not\perp a$, then there exists l such that $x - l \cdot b \perp a$.

(10) If $a \perp b + c$ and $b \perp c + a$, then $c \perp a + b$.

(11) $\Theta_S \perp a$.

(12) If $a \perp b$, then $b \perp a$.

(13) If $a \not\perp b$ and $c + a \perp b$, then $c \not\perp b$.

(14) If $b \not\perp a$ and $c \perp a$, then $b + c \not\perp a$.

(15) If $b \not\perp a$ and $l \neq 0_S$, then $l \cdot b \not\perp a$ and $b \not\perp l \cdot a$.

(16) If $a \perp b$, then $-a \perp b$.

(17) If $a + b \perp c$ and $a \perp c$, then $b \perp c$.

(18) If $a + b \perp c$ and $b \perp c$, then $a \perp c$.

(19) If $a \not\perp c$, then $a + b \not\perp c$ or $(\Omega_S + \Omega_S) \cdot a + b \not\perp c$.

(20) If $a' \not\perp a$ and $a' \perp b$ and $b' \not\perp b$ and $b' \perp a$, then $a' + b' \not\perp a$ and $a' + b' \not\perp b$.

(21) If $a \neq \Theta_S$ and $b \neq \Theta_S$, then there exists p such that $p \not\perp a$ and $p \not\perp b$.

(22) If $\Omega_S + \Omega_S \neq 0_S$ and $a \neq \Theta_S$ and $b \neq \Theta_S$ and $c \neq \Theta_S$, then there exists p such that $p \not\perp a$ and $p \not\perp b$ and $p \not\perp c$.

(23) If $a - b \perp d$ and $a - c \perp d$, then $b - c \perp d$.

(24) If $b \not\perp a$ and $x - k \cdot b \perp a$ and $x - l \cdot b \perp a$, then $k = l$.

(25) If $\Omega_S + \Omega_S \neq 0_S$, then $a \perp a$.

Let us consider S, a, b, x. Let us assume that $b \not\perp a$. The functor $\mathrm{J}(a, b, x)$ yields an element of the carrier of the scalars of S and is defined by:

for every element l of the carrier of the scalars of S such that $x - l \cdot b \perp a$ holds $\mathrm{J}(a, b, x) = l$.

The following propositions are true:

(26) If $b \not\perp a$ and $x - l \cdot b \perp a$, then $\mathrm{J}(a, b, x) = l$.

(27) If $b \not\perp a$, then $x - \mathrm{J}(a, b, x) \cdot b \perp a$.

(28) If $b \not\perp a$, then $\mathrm{J}(a, b, l \cdot x) = l \cdot \mathrm{J}(a, b, x)$.

(29) If $b \not\perp a$, then $\mathrm{J}(a, b, x + y) = \mathrm{J}(a, b, x) + \mathrm{J}(a, b, y)$.

(30) If $b \not\perp a$ and $l \neq 0_S$, then $\mathrm{J}(a, l \cdot b, x) = l^{-1} \cdot \mathrm{J}(a, b, x)$.

(31) If $b \not\perp a$ and $l \neq 0_S$, then $\mathrm{J}(l \cdot a, b, x) = \mathrm{J}(a, b, x)$.

(32) If $b \not\perp a$ and $p \perp a$, then $\mathrm{J}(a, b + p, c) = \mathrm{J}(a, b, c)$ and $\mathrm{J}(a, b, c + p) = \mathrm{J}(a, b, c)$.

(33) If $b \not\perp a$ and $p \perp b$ and $p \perp c$, then $\mathrm{J}(a + p, b, c) = \mathrm{J}(a, b, c)$.

(34) If $b \not\perp a$ and $c - b \perp a$, then $\mathrm{J}(a, b, c) = \Omega_S$.

(35) If $b \not\perp a$, then $\mathrm{J}(a, b, b) = \Omega_S$.

(36) If $b \not\perp a$, then $x \perp a$ if and only if $\mathrm{J}(a, b, x) = 0_S$.

(37) If $b \not\perp a$ and $q \not\perp a$, then $\mathrm{J}(a, b, p) \cdot \mathrm{J}(a, b, q)^{-1} = \mathrm{J}(a, q, p)$.

(38) If $b \not\perp a$ and $c \not\perp a$, then $\mathrm{J}(a, b, c) = \mathrm{J}(a, c, b)^{-1}$.

(39) If $b \not\perp a$ and $b \perp c + a$, then $\mathrm{J}(a, b, c) = \mathrm{J}(c, b, a)$.

(40) If $a \not\perp b$ and $c \not\perp b$, then $\mathrm{J}(c, b, a) = (-\mathrm{J}(b, a, c)^{-1}) \cdot \mathrm{J}(a, b, c)$.

(41) If $\Omega_S + \Omega_S \neq 0_S$ and $a \not\perp p$ and $a \not\perp q$ and $b \not\perp p$ and $b \not\perp q$, then $\mathrm{J}(a, p, q) \cdot \mathrm{J}(b, q, p) = \mathrm{J}(p, a, b) \cdot \mathrm{J}(q, b, a)$.

(42) If $\Omega_S + \Omega_S \neq 0_S$ and $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$, then $\mathrm{J}(a,q,p) \cdot \mathrm{J}(p,a,x) = \mathrm{J}(x,q,p) \cdot \mathrm{J}(q,a,x)$.

(43) Suppose $\Omega_S + \Omega_S \neq 0_S$ and $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$ and $b \not\perp a$. Then $(\mathrm{J}(a,b,p) \cdot \mathrm{J}(p,a,x)) \cdot \mathrm{J}(x,p,y) = (\mathrm{J}(a,b,q) \cdot \mathrm{J}(q,a,x)) \cdot \mathrm{J}(x,q,y)$.

(44) If $a \not\perp p$ and $x \not\perp p$ and $y \not\perp p$, then $\mathrm{J}(p,a,x) \cdot \mathrm{J}(x,p,y) = (-\mathrm{J}(p,a,y)) \cdot \mathrm{J}(y,p,x)$.

Let us consider S, x, y, a, b. Let us assume that $b \not\perp a$ and $\Omega_S + \Omega_S \neq 0_S$. The functor $x \cdot_{a,b} y$ yields an element of the carrier of the scalars of S and is defined by:

for every q such that $q \not\perp a$ and $q \not\perp x$ holds $x \cdot_{a,b} y = (\mathrm{J}(a,b,q) \cdot \mathrm{J}(q,a,x)) \cdot \mathrm{J}(x,q,y)$ if there exists p such that $p \not\perp a$ and $p \not\perp x$, $x \cdot_{a,b} y = 0_S$ if for every p holds $p \perp a$ or $p \perp x$.

One can prove the following propositions:

(45) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$ and $p \not\perp a$ and $p \not\perp x$, then $x \cdot_{a,b} y = (\mathrm{J}(a,b,p) \cdot \mathrm{J}(p,a,x)) \cdot \mathrm{J}(x,p,y)$.

(46) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$ and for every p holds $p \perp a$ or $p \perp x$, then $x \cdot_{a,b} y = 0_S$.

(47) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$ and $x = \Theta_S$, then $x \cdot_{a,b} y = 0_S$.

(48) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$, then $x \cdot_{a,b} y = 0_S$ if and only if $y \perp x$.

(49) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$, then $x \cdot_{a,b} y = -y \cdot_{a,b} x$.

(50) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$, then $x \cdot_{a,b} (l \cdot y) = l \cdot x \cdot_{a,b} y$.

(51) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$, then $x \cdot_{a,b} (y + z) = x \cdot_{a,b} y + x \cdot_{a,b} z$.

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