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Introduction to Trees

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Summary. The article consists of two parts: the first one deals with the concept of the prefixes of a finite sequence, the second one introduces and deals with the concept of tree. Besides some auxiliary propositions concerning finite sequences are presented. The trees are introduced as non-empty sets of finite sequences of natural numbers which are closed on prefixes and on sequences of less numbers (i.e. if $\langle n_1, n_2, \ldots, n_k\rangle$ is a vertex (element) of a tree and $m_i \leq n_i$ for $i = 1, 2, \ldots, k$, then $\langle m_1, m_2, \ldots, m_k\rangle$ also is). Finite trees, elementary trees with n leaves, the leaves and the subtrees of a tree, the inserting of a tree into another tree, with a node used for detemining the place of insertion, antichains of prefixes, and height and width of finite trees are introduced.

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The notation and terminology used in this paper have been introduced in the following papers: [8], [7], [2], [5], [4], [6], [3], and [1]. For simplicity we adopt the following rules: D is a non-empty set, X is a set, x, y are arbitrary, k, n are natural numbers, and p, q, r are finite sequences of elements of $\mathbb{N}$. We now state several propositions:

(1) For all finite sequences p, q such that $q = p \restriction \operatorname{Seg} n$ holds $\operatorname{len} q \leq n$.

(2) For all finite sequences p, q such that $q = p \restriction \operatorname{Seg} n$ holds $\operatorname{len} q \leq \operatorname{len} p$.

(3) For all finite sequences p, r such that $r = p \restriction \operatorname{Seg} n$ there exists q being a finite sequence such that $p = r \frown q$.

(4) $\varepsilon \neq \langle x\rangle$.

(5) For all finite sequences p, q such that $p = p \frown q$ or $p = q \frown p$ holds $q = \varepsilon$.

(6) For all finite sequences p, q such that $p \frown q = \langle x\rangle$ holds $p = \langle x\rangle$ and $q = \varepsilon$ or $p = \varepsilon$ and $q = \langle x\rangle$.

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Let p, q be finite sequences. The predicate $p \preceq q$ is defined by:
there exists n such that $p = q \restriction \operatorname{Seg} n$.

We now state a number of propositions:

(7) For all finite sequences p, q holds $p \preceq q$ if and only if there exists n such that $p = q \restriction \operatorname{Seg} n$.

(8) For all finite sequences p, q holds $p \preceq q$ if and only if there exists r being a finite sequence such that $q = p \frown r$.

(9) For all finite sequences p, q such that $p \preceq q$ holds $\operatorname{len} p \leq \operatorname{len} q$.

(10) For every finite sequence p holds $\varepsilon \preceq p$ and $\varepsilon_D \preceq p$.

(11) For every finite sequence p such that $p \preceq \varepsilon$ holds $p = \varepsilon$.

(12) For every finite sequence p holds $p \preceq p$.

(13) For all finite sequences p, q such that $p \preceq q$ and $q \preceq p$ holds $p = q$.

(14) For all finite sequences p, q, r such that $p \preceq q$ and $q \preceq r$ holds $p \preceq r$.

(15) For all finite sequences p, q such that $p \preceq q$ and $\operatorname{len} p = \operatorname{len} q$ holds $p = q$.

(16) $\langle x \rangle \preceq \langle y \rangle$ if and only if $x = y$.

We now define two new predicates. Let p, q be finite sequences. The predicate p and q are comparable is defined by:
$p \preceq q$ or $q \preceq p$.
The predicate $p \prec q$ is defined by:
$p \preceq q$ and $p \neq q$.

One can prove the following propositions:

(17) For all finite sequences p, q holds p and q are comparable if and only if $p \preceq q$ or $q \preceq p$.

(18) For all finite sequences p, q holds $p \prec q$ if and only if $p \preceq q$ and $p \neq q$.

(19) For all finite sequences p, q such that p and q are comparable and $\operatorname{len} p = \operatorname{len} q$ holds $p = q$.

(20) For all finite sequences p, q holds $p \prec q$ or $p = q$ or $q \prec p$ if and only if p and q are comparable.

(21) For every finite sequence p holds p and p are comparable.

In the sequel p_1, p_2 will be finite sequences. Next we state a number of propositions:

(22) If p_1 and p_2 are comparable, then p_2 and p_1 are comparable.

(23) $\langle x \rangle$ and $\langle y \rangle$ are comparable if and only if $x = y$.

(24) For all finite sequences p, q such that $p \prec q$ holds $\operatorname{len} p < \operatorname{len} q$.

(25) For no finite sequence p holds $p \prec \varepsilon$ or $p \prec \varepsilon_D$.

(26) For no finite sequences p, q holds $p \prec q$ and $q \prec p$.

(27) For all finite sequences p, q, r such that $p \prec q$ and $q \prec r$ or $p \prec q$ and $q \preceq r$ or $p \preceq q$ and $q \prec r$ holds $p \prec r$.

(28) If $p_1 \preceq p_2$, then $p_2 \not\prec p_1$.

(29) If $p_1 \prec p_2$, then $p_2 \not\preceq p_1$.

(30) If $p_1 \frown \langle x \rangle \preceq p_2$, then $p_1 \prec p_2$.

(31) If $p_1 \preceq p_2$, then $p_1 \prec p_2 \frown \langle x \rangle$.

(32) If $p_1 \prec p_2 \frown \langle x \rangle$, then $p_1 \preceq p_2$.

(33) If $\varepsilon \prec p_2$ or $\varepsilon \neq p_2$, then $p_1 \prec p_1 \frown p_2$.

Let p be a finite sequence. The functor $\mathrm{Seg}_{\preceq}(p)$ yielding a set, is defined by:

$x \in \mathrm{Seg}_{\preceq}(p)$ if and only if there exists q being a finite sequence such that $x = q$ and $q \prec p$.

The following propositions are true:

(34) For every finite sequence p holds $X = \mathrm{Seg}_{\preceq}(p)$ if and only if for every x holds $x \in X$ if and only if there exists q being a finite sequence such that $x = q$ and $q \prec p$.

(35) For every finite sequence p such that $x \in \mathrm{Seg}_{\preceq}(p)$ holds x is a finite sequence.

(36) For all finite sequences p, q holds $p \in \mathrm{Seg}_{\preceq}(q)$ if and only if $p \prec q$.

(37) For all finite sequences p, q such that $p \in \mathrm{Seg}_{\preceq}(q)$ holds $\mathrm{len}\, p < \mathrm{len}\, q$.

(38) For all finite sequences p, q, r such that $q \frown r \in \mathrm{Seg}_{\preceq}(p)$ holds $q \in \mathrm{Seg}_{\preceq}(p)$.

(39) $\mathrm{Seg}_{\preceq}(\varepsilon) = \emptyset$.

(40) $\mathrm{Seg}_{\preceq}(\langle x \rangle) = \{\varepsilon\}$.

(41) For all finite sequences p, q such that $p \preceq q$ holds $\mathrm{Seg}_{\preceq}(p) \subseteq \mathrm{Seg}_{\preceq}(q)$.

(42) For all finite sequences p, q, r such that $q \in \mathrm{Seg}_{\preceq}(p)$ and $r \in \mathrm{Seg}_{\preceq}(p)$ holds q and r are comparable.

The mode tree, which widens to the type a non-empty set, is defined by:

it $\subseteq \mathbb{N}^*$ and for every p such that $p \in$ it holds $\mathrm{Seg}_{\preceq}(p) \subseteq$ it and for all p, k, n such that $p \frown \langle k \rangle \in$ it and $n \leq k$ holds $p \frown \langle n \rangle \in$ it.

Next we state a proposition

(43) D is a tree if and only if $D \subseteq \mathbb{N}^*$ and for every p such that $p \in D$ holds $\mathrm{Seg}_{\preceq}(p) \subseteq D$ and for all p, k, n such that $p \frown \langle k \rangle \in D$ and $n \leq k$ holds $p \frown \langle n \rangle \in D$.

In the sequel T, T_1 denote trees. The following proposition is true

(44) If $x \in T$, then x is a finite sequence of elements of $\mathbb{N}$.

Let us consider T. We see that it makes sense to consider the following mode for restricted scopes of arguments. Then all the objests of the mode element of T are a finite sequence of elements of $\mathbb{N}$.

The following propositions are true:

(45) For all finite sequences p, q such that $p \in T$ and $q \preceq p$ holds $q \in T$.

(46) For every finite sequence r such that $q \frown r \in T$ holds $q \in T$.

(47) $\varepsilon \in T$ and $\varepsilon_{\mathbb{N}} \in T$.

(48) $\{\varepsilon\}$ is a tree.

(49) $T \cup T_1$ is a tree.

(50) $T \cap T_1$ is a tree.

The mode finite tree, which widens to the type a tree, is defined by:

it is finite.

The following proposition is true

(51) T is a finite tree if and only if T is finite.

In the sequel fT, fT_1 will be finite trees. Next we state two propositions:

(52) $fT \cup fT_1$ is a finite tree.

(53) $fT \cap T$ is a finite tree and $T \cap fT$ is a finite tree.

Let us consider n. The functor elementary tree of n yielding a finite tree, is defined by:

elementary tree of $n = \{\langle k \rangle : k < n\} \cup \{\varepsilon\}$.

The following propositions are true:

(54) $fT =$ elementary tree of n if and only if $fT = \{\langle k \rangle : k < n\} \cup \{\varepsilon\}$.

(55) If $k < n$, then $\langle k \rangle \in$ elementary tree of n.

(56) elementary tree of $0 = \{\varepsilon\}$.

(57) If $p \in$ elementary tree of n, then $p = \varepsilon$ or there exists k such that $k < n$ and $p = \langle k \rangle$.

We now define two new functors. Let us consider T. The functor Leaves T yields a subset of T and is defined by:

$p \in$ Leaves T if and only if $p \in T$ and for no q holds $q \in T$ and $p \prec q$.

Let us consider p. Let us assume that $p \in T$. The functor $T \upharpoonright p$ yields a tree and is defined by:

$q \in T \upharpoonright p$ if and only if $p \frown q \in T$.

We now state three propositions:

(58) For every subset X of T holds $X =$ Leaves T if and only if for every p holds $p \in X$ if and only if $p \in T$ and for no q holds $q \in T$ and $p \prec q$.

(59) If $p \in T$, then $T_1 = T \upharpoonright p$ if and only if for every q holds $q \in T_1$ if and only if $p \frown q \in T$.

(60) $T \upharpoonright \varepsilon_{\mathbb{N}} = T$.

The arguments of the notions defined below are the following: T which is a finite tree; p which is an element of T. Then $T \upharpoonright p$ is a finite tree.

Let us consider T. Let us assume that Leaves $T \neq \emptyset$. The mode leaf of T, which widens to the type an element of T, is defined by:

it $\in$ Leaves T.

We now state a proposition

(61) If Leaves $T \neq \emptyset$, then for every element p of T holds p is a leaf of T if and only if $p \in$ Leaves T.

Let us consider T. The mode subtree of T, which widens to the type a tree, is defined by:

there exists p being an element of T such that it $= T \upharpoonright p$.

One can prove the following proposition

(62) T_1 is a subtree of T if and only if there exists p being an element of T such that $T_1 = T \upharpoonright p$.

In the sequel t is an element of T. Let us consider T, p, T_1. Let us assume that $p \in T$. The functor $T(p/T_1)$ yields a tree and is defined by:

$q \in T(p/T_1)$ if and only if $q \in T$ and $p \not\preceq q$ or there exists r such that $r \in T_1$ and $q = p \mathbin{^\frown} r$.

In the sequel T_2 is a tree. Next we state four propositions:

(63) If $p \in T_1$, then $T = T_1(p/T_2)$ if and only if for every q holds $q \in T$ if and only if $q \in T_1$ and $p \not\preceq q$ or there exists r such that $r \in T_2$ and $q = p \mathbin{^\frown} r$.

(64) If $p \in T$, then $T(p/T_1) = \{t_1 : p \not\preceq t_1\} \cup \{p \mathbin{^\frown} s : s = s\}$.

(65) If $p \in T$ and $q \in T_1$, then $p \mathbin{^\frown} q \in T(p/T_1)$.

(66) If $p \in T$, then $T_1 = (T(p/T_1)) \upharpoonright p$.

The arguments of the notions defined below are the following: T which is a finite tree; t which is an element of T; T_1 which is a finite tree. Then $T(t/T_1)$ is a finite tree.

In the sequel w will denote a finite sequence. The following two propositions are true:

(67) For every finite sequence p holds $\mathrm{Seg}_{\preceq}(p) \approx \mathrm{Seg}(\mathrm{len}\, p)$.

(68) For every finite sequence p holds $\mathrm{card}(\mathrm{Seg}_{\preceq}(p)) = \mathrm{len}\, p$.

The mode antichain of prefixes, which widens to the type a set, is defined by:

for every x such that $x \in$ it holds x is a finite sequence and for all p_1, p_2 such that $p_1 \in$ it and $p_2 \in$ it and $p_1 \neq p_2$ holds p_1 and p_2 are not comparable.

Next we state three propositions:

(69) X is an antichain of prefixes if and only if for every x such that $x \in X$ holds x is a finite sequence and for all p_1, p_2 such that $p_1 \in X$ and $p_2 \in X$ and $p_1 \neq p_2$ holds p_1 and p_2 are not comparable.

(70) $\{w\}$ is an antichain of prefixes.

(71) If p_1 and p_2 are not comparable, then $\{p_1, p_2\}$ is an antichain of prefixes.

Let us consider T. The mode antichain of prefixes of T, which widens to the type an antichain of prefixes, is defined by:

it $\subseteq T$.

We now state a proposition

(72) For every antichain S of prefixes holds S is an antichain of prefixes of T if and only if $S \subseteq T$.

In the sequel t_1, t_2 will be elements of T. The following three propositions are true:

(73) $\emptyset$ is an antichain of prefixes of T and $\{\varepsilon\}$ is an antichain of prefixes of T.

(74) $\{t\}$ is an antichain of prefixes of T.

(75) If t_1 and t_2 are not comparable, then $\{t_1, t_2\}$ is an antichain of prefixes of T.

We now define two new functors. Let T be a finite tree. The functor $\mathrm{height}\, T$ yields a natural number and is defined by:

there exists p such that $p \in T$ and $\mathrm{len}\, p = \mathrm{height}\, T$ and for every p such that $p \in T$ holds $\mathrm{len}\, p \leq \mathrm{height}\, T$.

The functor $\mathrm{width}\, T$ yielding a natural number, is defined by:

there exists X being an antichain of prefixes of T such that width $T = \text{card}\, X$ and for every antichain Y of prefixes of T holds $\text{card}\, Y \leq \text{card}\, X$.

We now state three propositions:

(76) For every finite tree T for every n holds $n = \text{height}\, T$ if and only if there exists p such that $p \in T$ and $\text{len}\, p = n$ and for every p such that $p \in T$ holds $\text{len}\, p \leq n$.

(77) For every finite tree T for every n holds $n = \text{width}\, T$ if and only if there exists X being an antichain of prefixes of T such that $n = \text{card}\, X$ and for every antichain Y of prefixes of T holds $\text{card}\, Y \leq \text{card}\, X$.

(78) $1 \leq \text{width}\, fT$.

The following propositions are true:

(79) $\text{height}(\text{elementary tree of } 0) = 0$.

(80) If $\text{height}\, fT = 0$, then $fT = \text{elementary tree of } 0$.

(81) $\text{height}(\text{elementary tree of}(n+1)) = 1$.

(82) $\text{width}(\text{elementary tree of } 0) = 1$.

(83) $\text{width}(\text{elementary tree of}(n+1)) = n + 1$.

(84) For every element t of fT holds $\text{height}(fT \restriction t) \leq \text{height}\, fT$.

(85) For every element t of fT such that $t \neq \varepsilon$ holds $\text{height}(fT \restriction t) < \text{height}\, fT$.

The scheme *Tree_Ind* deals with a unary predicate $\mathcal{P}$ and states that:

for every fT holds $\mathcal{P}[fT]$

provided the parameter satisfies the following condition:

- for every fT such that for every n such that $\langle n \rangle \in fT$ holds $\mathcal{P}[fT \restriction \langle n \rangle]$ holds $\mathcal{P}[fT]$.

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