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Zermelo Theorem and Axiom of Choice

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Summary. The article is continuation of [2] and [1], and the goal of it is show that Zermelo theorem (every set has a relation which well orders it - proposition (26)) and axiom of choice (for every non-empty family of non-empty and separate sets there is set which has exactly one common element with arbitraly family member - proposition (27)) are true. It is result of the Tarski's axiom A introduced in [5] and repeated in [6]. Inclusion as a settheoretical binary relation is introduced, the correspondence of well ordering relations to ordinal numbers is shown, and basic properties of equinumerosity are presented. Some facts are based on [4].

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The terminology and notation used in this paper are introduced in the following articles: [6], [7], [8], [3], [2], and [1]. For simplicity we adopt the following convention: X, Y, Z will denote sets, a will be arbitrary, R will denote a relation, and A, B will denote ordinal numbers. Let us consider X. The functor ${}^{\subseteq}{}_{X}$ yielding a relation, is defined by:

field ${}^{\subseteq}{}_{X} = X$ and for all Y, Z such that $Y \in X$ and $Z \in X$ holds $\langle Y, Z \rangle \in {}^{\subseteq}{}_{X}$ if and only if $Y \subseteq Z$.

The following propositions are true:

(1) $R = {}^{\subseteq}{}_{X}$ if and only if field $R = X$ and for all Y, Z such that $Y \in X$ and $Z \in X$ holds $\langle Y, Z \rangle \in R$ if and only if $Y \subseteq Z$.

(2) ${}^{\subseteq}{}_{X}$ is pseudo reflexive.

(3) ${}^{\subseteq}{}_{X}$ is transitive.

(4) ${}^{\subseteq}{}_{A}$ is connected.

(5) ${}^{\subseteq}{}_{X}$ is antisymmetric.

(6) ${}^{\subseteq}{}_{A}$ is well founded.

(7) ${}^{\subseteq}{}_{A}$ is well ordering relation.

(8) If $Y \subseteq X$, then ${}^{\subseteq}{}_{X} |^2 Y = {}^{\subseteq}{}_{Y}$.

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(9) For all A, X such that $X \subseteq A$ holds $\subseteq_X$ is well ordering relation.

We now state several propositions:

(10) If $A \in B$, then $A = \subseteq_B{-}\mathrm{Seg}(A)$.

(11) If $\subseteq_A$ and $\subseteq_B$ are isomorphic, then $A = B$.

(12) For all X, R, A, B such that R and $\subseteq_A$ are isomorphic and R and $\subseteq_B$ are isomorphic holds $A = B$.

(13) For every R such that R is well ordering relation and for every a such that $a \in \mathrm{field}\, R$ there exists A such that $R\,|^2\, R{-}\mathrm{Seg}(a)$ and $\subseteq_A$ are isomorphic there exists A such that R and $\subseteq_A$ are isomorphic.

(14) For every R such that R is well ordering relation there exists A such that R and $\subseteq_A$ are isomorphic.

Let us consider R. Let us assume that R is well ordering relation. The functor $\overline{R}$ yields an ordinal number and is defined by:

R and $\subseteq_{\overline{R}}$ are isomorphic.

Let us consider A, R. The predicate A is an order type of R is defined by:

$A = \overline{R}$.

One can prove the following propositions:

(15) If R is well ordering relation, then for every A holds $A = \overline{R}$ if and only if R and $\subseteq_A$ are isomorphic.

(16) A is an order type of R if and only if $A = \overline{R}$.

(17) If $X \subseteq A$, then $\overline{\subseteq_X} \subseteq A$.

We follow a convention: f will be a function and x, y, z, u will be arbitrary. One can prove the following proposition

(18) $X \approx Y$ if and only if there exists Z such that for every x such that $x \in X$ there exists y such that $y \in Y$ and $\langle x, y \rangle \in Z$ and for every y such that $y \in Y$ there exists x such that $x \in X$ and $\langle x, y \rangle \in Z$ and for all x, y, z, u such that $\langle x, y \rangle \in Z$ and $\langle z, u \rangle \in Z$ holds $x = z$ if and only if $y = u$.

Let us consider X, Y. Let us note that one can characterize the predicate $X \approx Y$ by the following (equivalent) condition: there exists f such that f is one-to-one and $\mathrm{dom}\, f = X$ and $\mathrm{rng}\, f = Y$.

Next we state several propositions:

(19) $X \approx Y$ if and only if there exists f such that f is one-to-one and $\mathrm{dom}\, f = X$ and $\mathrm{rng}\, f = Y$.

(20) $X \approx X$.

(21) If $X \approx Y$, then $Y \approx X$.

(22) If $X \approx Y$ and $Y \approx Z$, then $X \approx Z$.

(23) If R is well ordering relation and $X \approx \mathrm{field}\, R$, then there exists R such that R well orders X.

(24) If R is well ordering relation and $X \approx Y$ and $Y \subseteq \mathrm{field}\, R$, then there exists R such that R well orders X.

(25) If R well orders X, then $\mathrm{field}(R\,|^2\, X) = X$ and $R\,|^2\, X$ is well ordering relation.

(26) For every X there exists R such that R well orders X.

In the sequel M will be a non-empty family of sets. We now state a proposition

(27) If for every X such that $X \in M$ holds $X \neq \emptyset$ and for all X, Y such that $X \in M$ and $Y \in M$ and $X \neq Y$ holds $X \cap Y = \emptyset$, then there exists $Choice$ being a set such that for every X such that $X \in M$ there exists x such that $Choice \cap X = \{x\}$.

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