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Parallelity and Lines in Affine Spaces [1]

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Summary. In the article we introduce basic notions concerning affine spaces and investigate their fundamental properties. We define the function which to every nondegenerate pair of points assigns the line joining them and we extend the relation of parallelity to a relation between segments and lines, and between lines.

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The papers [3], [1], and [2] provide the notation and terminology for this paper. We adopt the following convention: AS will be an affine space and a, a', b, b', c, d, o, p, q, x, y, z, t, u, w will be elements of the points of AS. One can prove the following propositions:

(1) There exist elements x, y of the points of AS such that $x \neq y$.

(2) $x, y \parallel y, x$ and $x, y \parallel z, z$.

(3) If $x \neq y$ and $x, y \parallel z, t$ and $x, y \parallel u, w$, then $z, t \parallel u, w$.

(4) If $x, y \parallel x, z$, then $y, x \parallel y, z$.

(5) There exist x, y, z such that $x, y \nparallel x, z$.

(6) There exists t such that $x, z \parallel y, t$ and $y \neq t$.

(7) There exists t such that $x, y \parallel z, t$ and $x, z \parallel y, t$.

(8) If $z, x \parallel x, t$ and $x \neq z$, then there exists u such that $y, x \parallel x, u$ and $y, z \parallel t, u$.

Let us consider AS, a, b, c. The predicate $\mathbf{L}(a, b, c)$ is defined as follows:
$a, b \parallel a, c$.

The following propositions are true:

(9) $\mathbf{L}(a, b, c)$ if and only if $a, b \parallel a, c$.

(10) For every a there exists b such that $a \neq b$.

(11) $x, y \parallel y, x$ and $x, y \parallel x, y$.

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(12) $x, y \parallel z, z$ and $z, z \parallel x, y$.

(13) If $x, y \parallel z, t$, then $x, y \parallel t, z$ and $y, x \parallel z, t$ and $y, x \parallel t, z$ and $z, t \parallel x, y$ and $z, t \parallel y, x$ and $t, z \parallel x, y$ and $t, z \parallel y, x$.

(14) Suppose that
  (i) $a \neq b$,
  (ii) $a, b \parallel x, y$ and $a, b \parallel z, t$ or $a, b \parallel x, y$ and $z, t \parallel a, b$ or $x, y \parallel a, b$ and $z, t \parallel a, b$ or $x, y \parallel a, b$ and $a, b \parallel z, t$.
  Then $x, y \parallel z, t$.

(15) If $\mathbf{L}(x, y, z)$, then $\mathbf{L}(x, z, y)$ and $\mathbf{L}(y, x, z)$ and $\mathbf{L}(y, z, x)$ and $\mathbf{L}(z, x, y)$ and $\mathbf{L}(z, y, x)$.

(16) $\mathbf{L}(x, x, y)$ and $\mathbf{L}(x, y, y)$ and $\mathbf{L}(x, y, x)$.

(17) If $x \neq y$ and $\mathbf{L}(x, y, z)$ and $\mathbf{L}(x, y, t)$ and $\mathbf{L}(x, y, u)$, then $\mathbf{L}(z, t, u)$.

(18) If $x \neq y$ and $\mathbf{L}(x, y, z)$ and $x, y \parallel z, t$, then $\mathbf{L}(x, y, t)$.

(19) If $\mathbf{L}(x, y, z)$ and $\mathbf{L}(x, y, t)$, then $x, y \parallel z, t$.

(20) If $u \neq z$ and $\mathbf{L}(x, y, u)$ and $\mathbf{L}(x, y, z)$ and $\mathbf{L}(u, z, w)$, then $\mathbf{L}(x, y, w)$.

(21) There exist x, y, z such that not $\mathbf{L}(x, y, z)$.

(22) If $x \neq y$, then there exists z such that not $\mathbf{L}(x, y, z)$.

(23) If not $\mathbf{L}(o, a, b)$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a, b'$, then $b = b'$.

Let us consider AS, a, b. The functor $\mathrm{Line}(a, b)$ yielding a subset of the points of AS, is defined as follows:

for every x holds $x \in \mathrm{Line}(a, b)$ if and only if $\mathbf{L}(a, b, x)$.

In the sequel A, C, D, K are subsets of the points of AS. We now state several propositions:

(24) $A = \mathrm{Line}(a, b)$ if and only if for every x holds $x \in A$ if and only if $\mathbf{L}(a, b, x)$.

(25) $\mathrm{Line}(a, b) = \mathrm{Line}(b, a)$.

(26) $a \in \mathrm{Line}(a, b)$ and $b \in \mathrm{Line}(a, b)$.

(27) If $c \in \mathrm{Line}(a, b)$ and $d \in \mathrm{Line}(a, b)$ and $c \neq d$, then $\mathrm{Line}(c, d) \subseteq \mathrm{Line}(a, b)$.

(28) If $c \in \mathrm{Line}(a, b)$ and $d \in \mathrm{Line}(a, b)$ and $a \neq b$, then $\mathrm{Line}(a, b) \subseteq \mathrm{Line}(c, d)$.

Let us consider AS, A. We say that A is a line if and only if:

there exist a, b such that $a \neq b$ and $A = \mathrm{Line}(a, b)$.

One can prove the following propositions:

(29) A is a line if and only if there exist a, b such that $a \neq b$ and $A = \mathrm{Line}(a, b)$.

(30) For all a, b, A, C such that A is a line and C is a line and $a \in A$ and $b \in A$ and $a \in C$ and $b \in C$ holds $a = b$ or $A = C$.

(31) If A is a line, then there exist a, b such that $a \in A$ and $b \in A$ and $a \neq b$.

(32) If A is a line and $a \in A$, then there exists b such that $a \neq b$ and $b \in A$.

(33) $\mathbf{L}(a,b,c)$ if and only if there exists A such that A is a line and $a \in A$ and $b \in A$ and $c \in A$.

Let us consider AS, a, b, A. The predicate $a,b \parallel A$ is defined by:
there exist c, d such that $c \neq d$ and $A = \mathrm{Line}(c,d)$ and $a,b \parallel c,d$.

The following proposition is true

(34) $a,b \parallel A$ if and only if there exist c, d such that $c \neq d$ and $A = \mathrm{Line}(c,d)$ and $a,b \parallel c,d$.

Let us consider AS, A, C. The predicate $A \parallel C$ is defined as follows:
there exist a, b such that $A = \mathrm{Line}(a,b)$ and $a \neq b$ and $a,b \parallel C$.

We now state a number of propositions:

(35) $A \parallel C$ if and only if there exist a, b such that $A = \mathrm{Line}(a,b)$ and $a \neq b$ and $a,b \parallel C$.

(36) If $c \in \mathrm{Line}(a,b)$ and $a \neq b$, then $d \in \mathrm{Line}(a,b)$ if and only if $a,b \parallel c,d$.

(37) If A is a line and $a \in A$, then $b \in A$ if and only if $a,b \parallel A$.

(38) $a \neq b$ and $A = \mathrm{Line}(a,b)$ if and only if A is a line and $a \in A$ and $b \in A$ and $a \neq b$.

(39) If A is a line and $a \in A$ and $b \in A$ and $a \neq b$ and $\mathbf{L}(a,b,x)$, then $x \in A$.

(40) If there exist a, b such that $a,b \parallel A$, then A is a line.

(41) If $c \in A$ and $d \in A$ and A is a line and $c \neq d$, then $a,b \parallel A$ if and only if $a,b \parallel c,d$.

(42) If $c \neq d$ and $a,b \parallel c,d$, then $a,b \parallel \mathrm{Line}(c,d)$.

(43) If $a \neq b$, then $a,b \parallel \mathrm{Line}(a,b)$.

(44) If A is a line, then $a,b \parallel A$ if and only if there exist c, d such that $c \neq d$ and $c \in A$ and $d \in A$ and $a,b \parallel c,d$.

(45) If A is a line and $a,b \parallel A$ and $c,d \parallel A$, then $a,b \parallel c,d$.

(46) If $a,b \parallel A$ and $a,b \parallel p,q$ and $a \neq b$, then $p,q \parallel A$.

(47) If A is a line, then $a,a \parallel A$.

(48) If $a,b \parallel A$, then $b,a \parallel A$.

(49) If $a,b \parallel A$ and $a \notin A$, then $b \notin A$.

(50) If $A \parallel C$, then A is a line and C is a line.

(51) $A \parallel C$ if and only if there exist a, b, c, d such that $a \neq b$ and $c \neq d$ and $a,b \parallel c,d$ and $A = \mathrm{Line}(a,b)$ and $C = \mathrm{Line}(c,d)$.

(52) If A is a line and C is a line and $a \in A$ and $b \in A$ and $c \in C$ and $d \in C$ and $a \neq b$ and $c \neq d$, then $A \parallel C$ if and only if $a,b \parallel c,d$.

(53) If $a \in A$ and $b \in A$ and $c \in C$ and $d \in C$ and $A \parallel C$, then $a,b \parallel c,d$.

(54) If $a \in A$ and $b \in A$ and $A \parallel C$, then $a,b \parallel C$.

(55) If A is a line, then $A \parallel A$.

(56) If $A \parallel C$, then $C \parallel A$.

(57) If $a,b \parallel A$ and $A \parallel C$, then $a,b \parallel C$.

(58) If $A \parallel C$ and $C \parallel D$ or $A \parallel C$ and $D \parallel C$ or $C \parallel A$ and $C \parallel D$ or $C \parallel A$ and $D \parallel C$, then $A \parallel D$.

(59) If $A \parallel C$ and $p \in A$ and $p \in C$, then $A = C$.

(60) If $x \in K$ and $a \notin K$ and $a, b \parallel K$, then $a = b$ or not $\mathbf{L}(x, a, b)$.

(61) If $a, b \parallel K$ and $a', b' \parallel K$ and $\mathbf{L}(p, a, a')$ and $\mathbf{L}(p, b, b')$ and $p \in K$ and $a \notin K$ and $a = b$, then $a' = b'$.

(62) If A is a line and $a \in A$ and $b \in A$ and $c \in A$ and $a \neq b$ and $a, b \parallel c, d$, then $d \in A$.

(63) For all a, A such that A is a line there exists C such that $a \in C$ and $A \parallel C$.

(64) If $A \parallel C$ and $A \parallel D$ and $p \in C$ and $p \in D$, then $C = D$.

(65) If A is a line and $a \in A$ and $b \in A$ and $c \in A$ and $d \in A$, then $a, b \parallel c, d$.

(66) If A is a line and $a \in A$ and $b \in A$, then $a, b \parallel A$.

(67) If $a, b \parallel A$ and $a, b \parallel C$ and $a \neq b$, then $A \parallel C$.

(68) If not $\mathbf{L}(o, a, b)$ and $\mathbf{L}(o, a, a')$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a', b'$ and $a' = b'$, then $a' = o$ and $b' = o$.

(69) If not $\mathbf{L}(o, a, b)$ and $\mathbf{L}(o, a, a')$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a', b'$ and $a' = o$, then $b' = o$.

(70) If not $\mathbf{L}(o, a, b)$ and $\mathbf{L}(o, a, a')$ and $\mathbf{L}(o, b, b')$ and $\mathbf{L}(o, b, x)$ and $a, b \parallel a', b'$ and $a, b \parallel a', x$, then $b' = x$.

(71) For all a, b, A such that A is a line and $a \in A$ and $b \in A$ and $a \neq b$ holds $A = \mathrm{Line}(a, b)$.

We adopt the following convention: AP will be an affine plane, a, b, c, d, x, p will be elements of the points of AP, and A, C will be subsets of the points of AP. One can prove the following three propositions:

(72) If A is a line and C is a line and $A \nparallel C$, then there exists x such that $x \in A$ and $x \in C$.

(73) If A is a line and $a, b \nparallel A$, then there exists x such that $x \in A$ and $\mathbf{L}(a, b, x)$.

(74) If $a, b \nparallel c, d$, then there exists p such that $\mathbf{L}(a, b, p)$ and $\mathbf{L}(c, d, p)$.

References

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