FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Analytical Ordered Affine Spaces [1]

Henryk Oryszczyszyn
Warsaw University
Białystok

Krzysztof Prażmowski
Warsaw University
Białystok

Summary. In the article with a given arbitrary real linear space we correlate the (ordered) affine space defined in terms of a directed parallelity of segments. The abstract contains a construction of the ordered affine structure associated with a vector space; this is a structure of the type which frequently occurs in geometry and consists of the set of points and a binary relation on segments. For suitable underlying vector spaces we prove that the corresponding affine structures are ordered affine spaces or ordered affine planes, i.e. that they satisfy appropriate axioms. A formal definition of an arbitrary ordered affine space and an arbitrary ordered affine plane is given.

MML Identifier: ANALOAF.

The notation and terminology used here have been introduced in the following articles: [4], [3], [2], [1], and [5]. We adopt the following rules: V will denote a real linear space, p, q, u, v, w, y will denote vectors of V, and a, b will denote real numbers. Let us consider V, u, v, w, y. The predicate $u,v \upuparrows w,y$ is defined by:

$u = v$ or $w = y$ or there exist a, b such that $0 < a$ and $0 < b$ and $a \cdot (v-u) = b \cdot (y-w)$.

Next we state a number of propositions:

(1) $u,v \upuparrows w,y$ if and only if $u = v$ or $w = y$ or there exist a, b such that $0 < a$ and $0 < b$ and $a \cdot (v-u) = b \cdot (y-w)$.

(2) If $0 < a$ and $0 < b$, then $0 < a+b$.

(3) If $a \neq b$, then $0 < a-b$ or $0 < b-a$.

(4) $(w-v)+(v-u) = w-u$.

(5) $-(u-v) = v-u$.

(6) $w-(u-v) = w+(v-u)$.

(7) $(w-u)+u = w$.

[1]Supported by RPBP.III.24-C6

ISSN 0777–4028

(8) $(w + u) - u = w$.

(9) If $y + u = v + w$, then $y - w = v - u$.

(10) $a \cdot (u - v) = -a \cdot (v - u)$.

(11) $(a - b) \cdot (u - v) = (b - a) \cdot (v - u)$.

(12) If $a \neq 0$ and $a \cdot u = v$, then $u = a^{-1} \cdot v$.

(13) If $a \neq 0$ and $a \cdot u = v$, then $u = a^{-1} \cdot v$ but if $a \neq 0$ and $u = a^{-1} \cdot v$, then $a \cdot u = v$.

(14) If $u = v$ or $w = y$, then $u, v \upuparrows w, y$.

(15) If $a \cdot (v - u) = b \cdot (y - w)$ and $0 < a$ and $0 < b$, then $u, v \upuparrows w, y$.

(16) If $u, v \upuparrows w, y$ and $u \neq v$ and $w \neq y$, then there exist a, b such that $a \cdot (v - u) = b \cdot (y - w)$ and $0 < a$ and $0 < b$.

(17) $u, v \upuparrows u, v$.

(18) $u, v \upuparrows w, w$ and $u, u \upuparrows v, w$.

(19) If $u, v \upuparrows v, u$, then $u = v$.

(20) If $p \neq q$ and $p, q \upuparrows u, v$ and $p, q \upuparrows w, y$, then $u, v \upuparrows w, y$.

(21) If $u, v \upuparrows w, y$, then $v, u \upuparrows y, w$ and $w, y \upuparrows u, v$.

(22) If $u, v \upuparrows v, w$, then $u, v \upuparrows u, w$.

(23) If $u, v \upuparrows u, w$, then $u, v \upuparrows v, w$ or $u, w \upuparrows w, v$.

(24) If $v - u = y - w$, then $u, v \upuparrows w, y$.

(25) If $y = (v + w) - u$, then $u, v \upuparrows w, y$ and $u, w \upuparrows v, y$.

(26) If there exist p, q such that $p \neq q$, then for every u, v, w there exists y such that $u, v \upuparrows w, y$ and $u, w \upuparrows v, y$ and $v \neq y$.

(27) If $p \neq v$ and $v, p \upuparrows p, w$, then there exists y such that $u, p \upuparrows p, y$ and $u, v \upuparrows w, y$.

(28) If for all a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$, then $u \neq v$ and $u \neq 0_V$ and $v \neq 0_V$.

(29) If there exist u, v such that for all a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$, then there exist u, v, w, y such that $u, v \not\upuparrows w, y$ and $u, v \not\upuparrows y, w$.

(30) If $a - b = 0$, then $a = b$.

Next we state a proposition

(31) Suppose there exist p, q such that for every w there exist a, b such that $a \cdot p + b \cdot q = w$. Then for all u, v, w, y such that $u, v \not\upuparrows w, y$ and $u, v \not\upuparrows y, w$ there exists a vector z of V such that $u, v \upuparrows u, z$ or $u, v \upuparrows z, u$ but $w, y \upuparrows w, z$ or $w, y \upuparrows z, w$.

We consider affine structures which are systems

⟨ points, a congruence ⟩

where the points is a non-empty set and the congruence is a relation on [: the points, the points :]. We adopt the following convention: AS will denote an affine structure and a, b, c, d will denote elements of the points of AS. Let us consider AS, a, b, c, d. The predicate $a, b \upuparrows c, d$ is defined by:

$\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of AS.

We now state a proposition

(32) $a, b \upuparrows c, d$ if and only if $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of AS.

In the sequel x, z are arbitrary. Let us consider V. The functor $\upuparrows_V$ yields a relation on $[\!:$ the vectors of V, the vectors of $V :\!]$ and is defined as follows:

$\langle x, z\rangle \in \upuparrows_V$ if and only if there exist u, v, w, y such that $x = \langle u, v\rangle$ and $z = \langle w, y\rangle$ and $u, v \upuparrows w, y$.

One can prove the following proposition

(33) $\langle\langle u, v\rangle, \langle w, y\rangle\rangle \in \upuparrows_V$ if and only if $u, v \upuparrows w, y$.

Let us consider V. The functor OASpace V yields an affine structure and is defined as follows:

OASpace $V = \langle$ the vectors of $V, \upuparrows_V\rangle$.

Next we state three propositions:

(34) OASpace $V = \langle$ the vectors of $V, \upuparrows_V\rangle$.

(35) Suppose there exist u, v such that for all real numbers a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$. Then

- (i) there exist elements a, b of the points of OASpace V such that $a \neq b$,
- (ii) for all elements a, b, c, d, p, q, r, s of the points of OASpace V holds $a, b \upuparrows c, c$ but if $a, b \upuparrows b, a$, then $a = b$ but if $a \neq b$ and $a, b \upuparrows p, q$ and $a, b \upuparrows r, s$, then $p, q \upuparrows r, s$ but if $a, b \upuparrows c, d$, then $b, a \upuparrows d, c$ but if $a, b \upuparrows b, c$, then $a, b \upuparrows a, c$ but if $a, b \upuparrows a, c$, then $a, b \upuparrows b, c$ or $a, c \upuparrows c, b$,
- (iii) there exist elements a, b, c, d of the points of OASpace V such that $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$,
- (iv) for every elements a, b, c of the points of OASpace V there exists an element d of the points of OASpace V such that $a, b \upuparrows c, d$ and $a, c \upuparrows b, d$ and $b \neq d$,
- (v) for all elements p, a, b, c of the points of OASpace V such that $p \neq b$ and $b, p \upuparrows p, c$ there exists an element d of the points of OASpace V such that $a, p \upuparrows p, d$ and $a, b \upuparrows c, d$.

(36) Suppose there exist vectors p, q of V such that for every vector w of V there exist real numbers a, b such that $a \cdot p + b \cdot q = w$. Let a, b, c, d be elements of the points of OASpace V. Then if $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$, then there exists an element t of the points of OASpace V such that $a, b \upuparrows a, t$ or $a, b \upuparrows t, a$ but $c, d \upuparrows c, t$ or $c, d \upuparrows t, c$.

An affine structure is called an ordered affine space if:

(i) there exist elements a, b of the points of it such that $a \neq b$,

(ii) for all elements a, b, c, d, p, q, r, s of the points of it holds $a, b \upuparrows c, c$ but if $a, b \upuparrows b, a$, then $a = b$ but if $a \neq b$ and $a, b \upuparrows p, q$ and $a, b \upuparrows r, s$, then $p, q \upuparrows r, s$ but if $a, b \upuparrows c, d$, then $b, a \upuparrows d, c$ but if $a, b \upuparrows b, c$, then $a, b \upuparrows a, c$ but if $a, b \upuparrows a, c$, then $a, b \upuparrows b, c$ or $a, c \upuparrows c, b$,

(iii) there exist elements a, b, c, d of the points of it such that $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$,

(iv) for every elements a, b, c of the points of it there exists an element d of the points of it such that $a, b \upuparrows c, d$ and $a, c \upuparrows b, d$ and $b \neq d$,
(v) for all elements p, a, b, c of the points of it such that $p \neq b$ and $b, p \upuparrows p, c$ there exists an element d of the points of it such that $a, p \upuparrows p, d$ and $a, b \upuparrows c, d$.

One can prove the following propositions:

(37) The following conditions are equivalent:

(i) there exist elements a, b of the points of AS such that $a \neq b$ and for all elements a, b, c, d, p, q, r, s of the points of AS holds $a, b \upuparrows c, c$ but if $a, b \upuparrows b, a$, then $a = b$ but if $a \neq b$ and $a, b \upuparrows p, q$ and $a, b \upuparrows r, s$, then $p, q \upuparrows r, s$ but if $a, b \upuparrows c, d$, then $b, a \upuparrows d, c$ but if $a, b \upuparrows b, c$, then $a, b \upuparrows a, c$ but if $a, b \upuparrows a, c$, then $a, b \upuparrows b, c$ or $a, c \upuparrows c, b$ and there exist elements a, b, c, d of the points of AS such that $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$ and for every elements a, b, c of the points of AS there exists an element d of the points of AS such that $a, b \upuparrows c, d$ and $a, c \upuparrows b, d$ and $b \neq d$ and for all elements p, a, b, c of the points of AS such that $p \neq b$ and $b, p \upuparrows p, c$ there exists an element d of the points of AS such that $a, p \upuparrows p, d$ and $a, b \upuparrows c, d$,

(ii) AS is an ordered affine space.

(38) If there exist u, v such that for all real numbers a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$, then $\mathrm{OASpace}\, V$ is an ordered affine space.

We adopt the following rules: A will denote an ordered affine space and a, b, c, d, p, q, r, s will denote elements of the points of A. We now state a number of propositions:

(39) There exist a, b such that $a \neq b$.

(40) $a, b \upuparrows c, c$.

(41) If $a, b \upuparrows b, a$, then $a = b$.

(42) If $a \neq b$ and $a, b \upuparrows p, q$ and $a, b \upuparrows r, s$, then $p, q \upuparrows r, s$.

(43) If $a, b \upuparrows c, d$, then $b, a \upuparrows d, c$.

(44) If $a, b \upuparrows b, c$, then $a, b \upuparrows a, c$.

(45) If $a, b \upuparrows a, c$, then $a, b \upuparrows b, c$ or $a, c \upuparrows c, b$.

(46) There exist a, b, c, d such that $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$.

(47) There exists d such that $a, b \upuparrows c, d$ and $a, c \upuparrows b, d$ and $b \neq d$.

(48) If $p \neq b$ and $b, p \upuparrows p, c$, then there exists d such that $a, p \upuparrows p, d$ and $a, b \upuparrows c, d$.

An ordered affine space is said to be an ordered affine plane if:

Let a, b, c, d be elements of the points of it . Then if $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$, then there exists an element p of the points of it such that $a, b \upuparrows a, p$ or $a, b \upuparrows p, a$ but $c, d \upuparrows c, p$ or $c, d \upuparrows p, c$.

We now state three propositions:

(49) The following conditions are equivalent:

(i) for all elements a, b, c, d of the points of A such that $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$ there exists an element p of the points of A such that $a, b \upuparrows a, p$ or $a, b \upuparrows p, a$ but $c, d \upuparrows c, p$ or $c, d \upuparrows p, c$,

(ii) A is an ordered affine plane.

(50) The following conditions are equivalent:

(i) there exist elements a, b of the points of AS such that $a \neq b$ and for all elements a, b, c, d, p, q, r, s of the points of AS holds $a, b \upuparrows c, c$ but if $a, b \upuparrows b, a$, then $a = b$ but if $a \neq b$ and $a, b \upuparrows p, q$ and $a, b \upuparrows r, s$, then $p, q \upuparrows r, s$ but if $a, b \upuparrows c, d$, then $b, a \upuparrows d, c$ but if $a, b \upuparrows b, c$, then $a, b \upuparrows a, c$ but if $a, b \upuparrows a, c$, then $a, b \upuparrows b, c$ or $a, c \upuparrows c, b$ and there exist elements a, b, c, d of the points of AS such that $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$ and for every elements a, b, c of the points of AS there exists an element d of the points of AS such that $a, b \upuparrows c, d$ and $a, c \upuparrows b, d$ and $b \neq d$ and for all elements p, a, b, c of the points of AS such that $p \neq b$ and $b, p \upuparrows p, c$ there exists an element d of the points of AS such that $a, p \upuparrows p, d$ and $a, b \upuparrows c, d$ and for all elements a, b, c, d of the points of AS such that $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$ there exists an element p of the points of AS such that $a, b \upuparrows a, p$ or $a, b \upuparrows p, a$ but $c, d \upuparrows c, p$ or $c, d \upuparrows p, c$,

(ii) AS is an ordered affine plane.

(51) If there exist u, v such that for all real numbers a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$ and for every w there exist real numbers a, b such that $w = a \cdot u + b \cdot v$, then OASpace V is an ordered affine plane.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[4] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[5] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

Received April 11, 1990