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Cardinal Arithmetics

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Summary. In the article addition, multiplication and power operation of cardinals are introduced. Presented are some properties of equipotence of Cartesian products, basic cardinal arithmetics laws (transformativity, associativity, distributivity), and some facts about finite sets.

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The articles [12], [11], [7], [8], [3], [4], [5], [10], [2], [6], [9], and [1] provide the terminology and notation for this paper. For simplicity we follow a convention: A, B denote ordinal numbers, K, M, N denote cardinal numbers, x, x_1, x_2, y, y_1, y_2 are arbitrary, X, Y, Z, X_1, X_2, Y_1, Y_2 denote sets, and f denotes a function. Let us consider x. The functor $[x]$ yielding a set, is defined by:

$[x] = x$.

Next we state several propositions:

(1) $[x] = x$.

(2) $\overline{\overline{X}} \leq \overline{\overline{Y}}$ if and only if there exists f such that $X = f \circ Y$ or $X \subseteq f \circ Y$.

(3) $\overline{\overline{f \circ X}} \leq \overline{\overline{X}}$.

(4) If $\overline{\overline{X}} < \overline{\overline{Y}}$, then $Y \setminus X \neq \emptyset$.

(5) If $x \in X$ and $X \approx Y$, then $Y \neq \emptyset$ and there exists x such that $x \in Y$.

(6) $2^X \approx 2^{\overline{\overline{X}}}$ and $\overline{\overline{2^X}} = \overline{\overline{2^{\overline{\overline{X}}}}}$.

(7) If $Z \in Y^X$, then $Z \approx X$ and $\overline{\overline{Z}} = \overline{\overline{X}}$.

We now define three new functors. Let us consider M, N. The functor $M + N$ yielding a cardinal number, is defined as follows:

$M + N = \overline{\overline{\mathrm{ord}(M) + \mathrm{ord}(N)}}$.

The functor $M \cdot N$ yielding a cardinal number, is defined by:

$M \cdot N = \overline{\overline{[: M, N :]}}$.

The functor M^N yielding a cardinal number, is defined by:

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$M^N = \overline{\overline{M^N}}$.

Next we state a number of propositions:

(8) $M + N = \overline{\overline{\mathrm{ord}(M) + \mathrm{ord}(N)}}$.

(9) $M \cdot N = \overline{\overline{[\!:M, N:\!]}}$.

(10) $M^N = \overline{\overline{M^N}}$.

(11) $[\!:X, Y:\!] \approx [\!:Y, X:\!]$ and $\overline{\overline{[\!:X, Y:\!]}} = \overline{\overline{[\!:Y, X:\!]}}$.

(12) $[\![\!:X, Y:\!], Z:\!] \approx [\!:X, [\!:Y, Z:\!]:\!]$ and $\overline{\overline{[\![\!:X, Y:\!], Z:\!]}} = \overline{\overline{[\!:X, [\!:Y, Z:\!]:\!]}}$.

(13) $X \approx [\!:X, \{x\}:\!]$ and $\overline{\overline{X}} = \overline{\overline{[\!:X, \{x\}:\!]}}$.

(14) (i) $[\!:X, Y:\!] \approx [\!:\overline{\overline{X}}, Y:\!]$,

(ii) $[\!:X, Y:\!] \approx [\!:X, \overline{\overline{Y}}:\!]$,

(iii) $[\!:X, Y:\!] \approx [\!:\overline{\overline{X}}, \overline{\overline{Y}}:\!]$,

(iv) $\overline{\overline{[\!:X, Y:\!]}} = \overline{\overline{[\!:\overline{\overline{X}}, Y:\!]}}$,

(v) $\overline{\overline{[\!:X, Y:\!]}} = \overline{\overline{[\!:X, \overline{\overline{Y}}:\!]}}$,

(vi) $\overline{\overline{[\!:X, Y:\!]}} = \overline{\overline{[\!:\overline{\overline{X}}, \overline{\overline{Y}}:\!]}}$.

(15) If $X_1 \approx Y_1$ and $X_2 \approx Y_2$, then $[\!:X_1, X_2:\!] \approx [\!:Y_1, Y_2:\!]$ and $\overline{\overline{[\!:X_1, X_2:\!]}} = \overline{\overline{[\!:Y_1, Y_2:\!]}}$.

(16) If $x_1 \neq x_2$, then $A + B \approx [\!:A, \{x_1\}:\!] \cup [\!:B, \{x_2\}:\!]$ and $\overline{\overline{A + B}} = \overline{\overline{[\!:A, \{x_1\}:\!] \cup [\!:B, \{x_2\}:\!]}}$.

(17) If $x_1 \neq x_2$, then $K + M \approx [\!:K, \{x_1\}:\!] \cup [\!:M, \{x_2\}:\!]$ and $K + M = \overline{\overline{[\!:K, \{x_1\}:\!] \cup [\!:M, \{x_2\}:\!]}}$.

(18) $A \cdot B \approx [\!:A, B:\!]$ and $\overline{\overline{A \cdot B}} = \overline{\overline{[\!:A, B:\!]}}$.

We now define three new functors. The cardinal number $\overline{\mathbf{0}}$ is defined by:

$\overline{\mathbf{0}} = \overline{\overline{\mathbf{0}}}$.

The cardinal number $\overline{\mathbf{1}}$ is defined as follows:

$\overline{\mathbf{1}} = \overline{\overline{\mathbf{1}}}$.

The cardinal number $\overline{\mathbf{2}}$ is defined as follows:

$\overline{\mathbf{2}} = \overline{\overline{\mathrm{succ}\,\mathbf{1}}}$.

The following propositions are true:

(19) $\overline{\mathbf{0}} = \overline{\overline{\mathbf{0}}}$ and $\overline{\mathbf{1}} = \overline{\overline{\mathbf{1}}}$ and $\overline{\mathbf{2}} = \overline{\overline{\mathrm{succ}\,\mathbf{1}}}$.

(20) $\overline{\mathbf{0}} = \mathbf{0}$ and $\overline{\mathbf{0}} = \emptyset$ and $\overline{\mathbf{1}} = \mathbf{1}$.

(21) $\overline{\mathbf{0}} = \overline{\overline{0}}$ and $\overline{\mathbf{1}} = \overline{\overline{1}}$ and $\overline{\mathbf{2}} = \overline{\overline{2}}$.

(22) $\overline{\mathbf{2}} = \{\mathbf{0}, \mathbf{1}\}$ and $\overline{\mathbf{2}} = \mathrm{succ}\,\mathbf{1}$.

(23) Suppose $X_1 \approx Y_1$ and $X_2 \approx Y_2$ and $x_1 \neq x_2$ and $y_1 \neq y_2$. Then $[\!:X_1, \{x_1\}:\!] \cup [\!:X_2, \{x_2\}:\!] \approx [\!:Y_1, \{y_1\}:\!] \cup [\!:Y_2, \{y_2\}:\!]$ and $\overline{\overline{[\!:X_1, \{x_1\}:\!] \cup [\!:X_2, \{x_2\}:\!]}} = \overline{\overline{[\!:Y_1, \{y_1\}:\!] \cup [\!:Y_2, \{y_2\}:\!]}}$.

(24) $\overline{\overline{A + B}} = \overline{\overline{A}} + \overline{\overline{B}}$.

(25) $\overline{\overline{A \cdot B}} = \overline{\overline{A}} \cdot \overline{\overline{B}}$.

(26) $[: X, \{0\} :] \cup [: Y, \{1\} :] \approx [: Y, \{0\} :] \cup [: X, \{1\} :]$ and $\overline{\overline{[: X, \{0\} :] \cup [: Y, \{1\} :]}} = \overline{\overline{[: Y, \{0\} :] \cup [: X, \{1\} :]}}$.

(27) $[: X_1, X_2 :] \cup [: Y_1, Y_2 :] \approx [: X_2, X_1 :] \cup [: Y_2, Y_1 :]$ and $\overline{\overline{[: X_1, X_2 :] \cup [: Y_1, Y_2 :]}} = \overline{\overline{[: X_2, X_1 :] \cup [: Y_2, Y_1 :]}}$.

(28) If $x \neq y$, then $\overline{\overline{X}} + \overline{\overline{Y}} = \overline{\overline{[: X, \{x\} :] \cup [: Y, \{y\} :]}}$.

(29) $M + \overline{\mathbf{0}} = M$ and $\overline{\mathbf{0}} + M = M$.

(30) $M + N = N + M$.

(31) $(K + M) + N = K + (M + N)$.

(32) $K \cdot \overline{\mathbf{0}} = \overline{\mathbf{0}}$ and $\overline{\mathbf{0}} \cdot K = \overline{\mathbf{0}}$.

(33) $K \cdot \overline{\mathbf{1}} = K$ and $\overline{\mathbf{1}} \cdot K = K$.

(34) $K \cdot M = M \cdot K$.

(35) $(K \cdot M) \cdot N = K \cdot (M \cdot N)$.

(36) $\overline{\mathbf{2}} \cdot K = K + K$ and $K \cdot \overline{\mathbf{2}} = K + K$.

(37) $K \cdot (M + N) = K \cdot M + K \cdot N$ and $(M + N) \cdot K = M \cdot K + N \cdot K$.

(38) $K^{\overline{\mathbf{0}}} = \overline{\mathbf{1}}$.

(39) If $K \neq \overline{\mathbf{0}}$, then $\overline{\mathbf{0}}^K = \overline{\mathbf{0}}$.

(40) $K^{\overline{\mathbf{1}}} = K$ and $\overline{\mathbf{1}}^K = \overline{\mathbf{1}}$.

(41) $K^{M+N} = (K^M) \cdot (K^N)$.

(42) $(K \cdot M)^N = (K^N) \cdot (M^N)$.

(43) $K^{M \cdot N} = (K^M)^N$.

(44) $\overline{\mathbf{2}}^{\overline{\overline{X}}} = \overline{\overline{2^X}}$.

(45) $K^{\overline{\mathbf{2}}} = K \cdot K$.

(46) $(K + M)^{\overline{\mathbf{2}}} = (K \cdot K + (\overline{\mathbf{2}} \cdot K) \cdot M) + M \cdot M$.

(47) $\overline{\overline{X \cup Y}} \leq \overline{\overline{X}} + \overline{\overline{Y}}$.

(48) If $X \cap Y = \emptyset$, then $\overline{\overline{X \cup Y}} = \overline{\overline{X}} + \overline{\overline{Y}}$.

In the sequel m, n will denote natural numbers. Next we state a number of propositions:

(49) $\mathrm{ord}(n + m) = \mathrm{ord}(n) + \mathrm{ord}(m)$.

(50) $\mathrm{ord}(n \cdot m) = \mathrm{ord}(n) \cdot \mathrm{ord}(m)$.

(51) $\overline{\overline{n + m}} = \overline{\overline{n}} + \overline{\overline{m}}$.

(52) $\overline{\overline{n \cdot m}} = \overline{\overline{n}} \cdot \overline{\overline{m}}$.

(53) If X is finite and Y is finite and $X \cap Y = \emptyset$, then $\mathrm{card}(X \cup Y) = \mathrm{card}\, X + \mathrm{card}\, Y$.

(54) If X is finite and $x \notin X$, then $\mathrm{card}(X \cup \{x\}) = \mathrm{card}\, X + 1$.

(55) If X is finite and Y is finite, then $\mathrm{card}\, X = \mathrm{card}\, Y$ if and only if $X \approx Y$.

(56) If X is finite and Y is finite, then $\overline{\overline{X}} = \overline{\overline{Y}}$ if and only if $\mathrm{card}\, X = \mathrm{card}\, Y$.

(57) If X is finite and Y is finite, then $\overline{\overline{X}} \leq \overline{\overline{Y}}$ if and only if $\operatorname{card} X \leq \operatorname{card} Y$.

(58) If X is finite and Y is finite, then $\overline{\overline{X}} < \overline{\overline{Y}}$ if and only if $\operatorname{card} X < \operatorname{card} Y$.

(59) If X is finite, then $X = \emptyset$ if and only if $\operatorname{card} X = 0$.

(60) If X is finite, then $\operatorname{card} X = 1$ if and only if there exists x such that $X = \{x\}$.

(61) If X is finite, then $X \approx \operatorname{ord}(\operatorname{card} X)$ and $X \approx \overline{\overline{\operatorname{card} X}}$ and $X \approx \operatorname{Seg}(\operatorname{card} X)$.

(62) If X is finite and Y is finite, then $\operatorname{card}(X \cup Y) \leq \operatorname{card} X + \operatorname{card} Y$.

(63) If $Y \subseteq X$ and X is finite, then $\operatorname{card}(X \setminus Y) = \operatorname{card} X - \operatorname{card} Y$.

(64) If X is finite and Y is finite, then $\operatorname{card}(X \cup Y) = (\operatorname{card} X + \operatorname{card} Y) - \operatorname{card}(X \cap Y)$.

(65) If X is finite and Y is finite, then $\operatorname{card}[\!: X, Y :\!] = \operatorname{card} X \cdot \operatorname{card} Y$.

(66) If $X \subseteq Y$ and Y is finite, then $\operatorname{card} X \leq \operatorname{card} Y$.

(67) If $X \subseteq Y$ and $X \neq Y$ and Y is finite, then $\operatorname{card} X < \operatorname{card} Y$ and $\overline{\overline{X}} < \overline{\overline{Y}}$.

(68) If $\overline{\overline{X}} \leq \overline{\overline{Y}}$ or $\overline{\overline{X}} < \overline{\overline{Y}}$ but Y is finite, then X is finite.

In the sequel x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 are arbitrary. One can prove the following propositions:

(69) $\operatorname{card}\{x_1, x_2\} \leq 2$.

(70) $\operatorname{card}\{x_1, x_2, x_3\} \leq 3$.

(71) $\operatorname{card}\{x_1, x_2, x_3, x_4\} \leq 4$.

(72) $\operatorname{card}\{x_1, x_2, x_3, x_4, x_5\} \leq 5$.

(73) $\operatorname{card}\{x_1, x_2, x_3, x_4, x_5, x_6\} \leq 6$.

(74) $\operatorname{card}\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \leq 7$.

(75) $\operatorname{card}\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \leq 8$.

(76) If $x_1 \neq x_2$, then $\operatorname{card}\{x_1, x_2\} = 2$.

(77) If $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$, then $\operatorname{card}\{x_1, x_2, x_3\} = 3$.

(78) If $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$, then $\operatorname{card}\{x_1, x_2, x_3, x_4\} = 4$.

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