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Tarski's Classes and Ranks

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Summary. In the article the Tarski's classes (non-empty families of sets satisfying Tarski's axiom A given in [9]) and the rank sets are introduced and some of their properties are shown. The transitive closure and the rank of a set is given here too.

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The terminology and notation used here have been introduced in the following articles: [9], [8], [7], [3], [4], [6], [5], [2], and [1]. For simplicity we adopt the following rules: W, X, Y, Z will denote sets, D will denote a non-empty set, f will denote a function, and x, y will be arbitrary. Let B be a set. We say that B is a Tarski-Class if and only if:

for all X, Y such that $X \in B$ and $Y \subseteq X$ holds $Y \in B$ and for every X such that $X \in B$ holds $2^X \in B$ and for every X such that $X \subseteq B$ holds $X \approx B$ or $X \in B$.

Let A, B be sets. We say that B is Tarski-Class of A if and only if:

$A \in B$ and B is a Tarski-Class.

Let A be a set. The functor $\mathbf{T}(A)$ yielding a non-empty family of sets, is defined as follows:

$\mathbf{T}(A)$ is Tarski-Class of A and for every D such that D is Tarski-Class of A holds $\mathbf{T}(A) \subseteq D$.

We now state several propositions:

(1) W is a Tarski-Class if and only if for all X, Y such that $X \in W$ and $Y \subseteq X$ holds $Y \in W$ and for every X such that $X \in W$ holds $2^X \in W$ and for every X such that $X \subseteq W$ holds $X \approx W$ or $X \in W$.

(2) W is a Tarski-Class if and only if for all X, Y such that $X \in W$ and $Y \subseteq X$ holds $Y \in W$ and for every X such that $X \in W$ holds $2^X \in W$ and for every X such that $X \subseteq W$ and $\overline{\overline{X}} < \overline{\overline{W}}$ holds $X \in W$.

(3) X is Tarski-Class of Y if and only if $Y \in X$ and X is a Tarski-Class.

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(4) For every non-empty family W of sets holds $W = \mathbf{T}(X)$ if and only if W is Tarski-Class of X and for every D such that D is Tarski-Class of X holds $W \subseteq D$.

(5) $X \in \mathbf{T}(X)$.

(6) If $Y \in \mathbf{T}(X)$ and $Z \subseteq Y$, then $Z \in \mathbf{T}(X)$.

(7) If $Y \in \mathbf{T}(X)$, then $2^Y \in \mathbf{T}(X)$.

(8) If $Y \subseteq \mathbf{T}(X)$, then $Y \approx \mathbf{T}(X)$ or $Y \in \mathbf{T}(X)$.

(9) If $Y \subseteq \mathbf{T}(X)$ and $\overline{\overline{Y}} < \overline{\overline{\mathbf{T}(X)}}$, then $Y \in \mathbf{T}(X)$.

We follow a convention: u, v will denote elements of $\mathbf{T}(X)$, A, B, C will denote ordinal numbers, and L, L_1 will denote transfinite sequences. Let us consider X, A. The functor $\mathbf{T}_A(X)$ is defined as follows:

there exists L such that $\mathbf{T}_A(X) = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\mathbf{0}) = \{X\}$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} A$ and $y = L(C)$ holds $L(\operatorname{succ} C) = (\{u : \bigvee_v [v \in [y] \wedge u \subseteq v]\} \cup \{2^v : v \in [y]\}) \cup 2^{[y]} \cap \mathbf{T}(X)$ and for all C, L_1 such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \bigcup(\operatorname{rng} L_1) \cap \mathbf{T}(X)$.

Let us consider X, A. Then $\mathbf{T}_A(X)$ is a subset of $\mathbf{T}(X)$.

Next we state a number of propositions:

(10) $\mathbf{T}_{\mathbf{0}}(X) = \{X\}$.

(11) $\mathbf{T}_{\operatorname{succ} A}(X) = (\{u : \bigvee_v [v \in \mathbf{T}_A(X) \wedge u \subseteq v]\} \cup \{2^v : v \in \mathbf{T}_A(X)\}) \cup 2^{\mathbf{T}_A(X)} \cap \mathbf{T}(X)$.

(12) If $A \neq \mathbf{0}$ and A is a limit ordinal number, then $\mathbf{T}_A(X) = \{u : \bigvee_B [B \in A \wedge u \in \mathbf{T}_B(X)]\}$.

(13) $Y \in \mathbf{T}_{\operatorname{succ} A}(X)$ if and only if $Y \subseteq \mathbf{T}_A(X)$ and $Y \in \mathbf{T}(X)$ or there exists Z such that $Z \in \mathbf{T}_A(X)$ but $Y \subseteq Z$ or $Y = 2^Z$.

(14) If $Y \subseteq Z$ and $Z \in \mathbf{T}_A(X)$, then $Y \in \mathbf{T}_{\operatorname{succ} A}(X)$.

(15) If $Y \in \mathbf{T}_A(X)$, then $2^Y \in \mathbf{T}_{\operatorname{succ} A}(X)$.

(16) If $A \neq \mathbf{0}$ and A is a limit ordinal number, then $x \in \mathbf{T}_A(X)$ if and only if there exists B such that $B \in A$ and $x \in \mathbf{T}_B(X)$.

(17) If $A \neq \mathbf{0}$ and A is a limit ordinal number and $Y \in \mathbf{T}_A(X)$ but $Z \subseteq Y$ or $Z = 2^Y$, then $Z \in \mathbf{T}_A(X)$.

(18) $\mathbf{T}_A(X) \subseteq \mathbf{T}_{\operatorname{succ} A}(X)$.

(19) If $A \subseteq B$, then $\mathbf{T}_A(X) \subseteq \mathbf{T}_B(X)$.

(20) There exists A such that $\mathbf{T}_A(X) = \mathbf{T}_{\operatorname{succ} A}(X)$.

(21) If $\mathbf{T}_A(X) = \mathbf{T}_{\operatorname{succ} A}(X)$, then $\mathbf{T}_A(X) = \mathbf{T}(X)$.

(22) There exists A such that $\mathbf{T}_A(X) = \mathbf{T}(X)$.

(23) There exists A such that $\mathbf{T}_A(X) = \mathbf{T}(X)$ and for every B such that $B \in A$ holds $\mathbf{T}_B(X) \neq \mathbf{T}(X)$.

(24) If $Y \neq X$ and $Y \in \mathbf{T}(X)$, then there exists A such that $Y \notin \mathbf{T}_A(X)$ and $Y \in \mathbf{T}_{\operatorname{succ} A}(X)$.

(25) If X is transitive, then for every A such that $A \neq \mathbf{0}$ holds $\mathbf{T}_A(X)$ is transitive.

(26) $\mathbf{T}_{\mathbf{0}}(X) \in \mathbf{T}_{\mathbf{1}}(X)$ and $\mathbf{T}_{\mathbf{0}}(X) \neq \mathbf{T}_{\mathbf{1}}(X)$.

(27) If X is transitive, then $\mathbf{T}(X)$ is transitive.

(28) If $Y \in \mathbf{T}(X)$, then $\overline{\overline{Y}} < \overline{\overline{\mathbf{T}(X)}}$.

(29) If $Y \in \mathbf{T}(X)$, then $Y \not\approx \mathbf{T}(X)$.

(30) If $x \in \mathbf{T}(X)$ and $y \in \mathbf{T}(X)$, then $\{x\} \in \mathbf{T}(X)$ and $\{x, y\} \in \mathbf{T}(X)$.

(31) If $x \in \mathbf{T}(X)$ and $y \in \mathbf{T}(X)$, then $\langle x, y \rangle \in \mathbf{T}(X)$.

(32) If $Y \subseteq \mathbf{T}(X)$ and $Z \subseteq \mathbf{T}(X)$, then $[\!: Y, Z :\!] \subseteq \mathbf{T}(X)$.

Let us consider A. The functor $\mathbf{R}_A$ is defined as follows:

there exists L such that $\mathbf{R}_A = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\mathbf{0}) = \emptyset$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} A$ and $y = L(C)$ holds $L(\operatorname{succ} C) = 2^{[y]}$ and for all C, L_1 such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \bigcup(\operatorname{rng} L_1)$.

Let us consider A. Then $\mathbf{R}_A$ is a set.

One can prove the following propositions:

(33) $\mathbf{R}_{\mathbf{0}} = \emptyset$.

(34) $\mathbf{R}_{\operatorname{succ} A} = 2^{\mathbf{R}_A}$.

(35) If $A \neq \mathbf{0}$ and A is a limit ordinal number, then for every x holds $x \in \mathbf{R}_A$ if and only if there exists B such that $B \in A$ and $x \in \mathbf{R}_B$.

(36) $X \subseteq \mathbf{R}_A$ if and only if $X \in \mathbf{R}_{\operatorname{succ} A}$.

(37) $\mathbf{R}_A$ is transitive.

(38) If $X \in \mathbf{R}_A$, then $X \subseteq \mathbf{R}_A$.

(39) $\mathbf{R}_A \subseteq \mathbf{R}_{\operatorname{succ} A}$.

(40) $\bigcup \mathbf{R}_A \subseteq \mathbf{R}_A$.

(41) If $X \in \mathbf{R}_A$, then $\bigcup X \in \mathbf{R}_A$.

(42) $A \in B$ if and only if $\mathbf{R}_A \in \mathbf{R}_B$.

(43) $A \subseteq B$ if and only if $\mathbf{R}_A \subseteq \mathbf{R}_B$.

(44) $A \subseteq \mathbf{R}_A$.

(45) For all A, X such that $X \in \mathbf{R}_A$ holds $X \not\approx \mathbf{R}_A$ and $\overline{\overline{X}} < \overline{\overline{\mathbf{R}_A}}$.

(46) $X \subseteq \mathbf{R}_A$ if and only if $2^X \subseteq \mathbf{R}_{\operatorname{succ} A}$.

(47) If $X \subseteq Y$ and $Y \in \mathbf{R}_A$, then $X \in \mathbf{R}_A$.

(48) $X \in \mathbf{R}_A$ if and only if $2^X \in \mathbf{R}_{\operatorname{succ} A}$.

(49) $x \in \mathbf{R}_A$ if and only if $\{x\} \in \mathbf{R}_{\operatorname{succ} A}$.

(50) $x \in \mathbf{R}_A$ and $y \in \mathbf{R}_A$ if and only if $\{x, y\} \in \mathbf{R}_{\operatorname{succ} A}$.

(51) $x \in \mathbf{R}_A$ and $y \in \mathbf{R}_A$ if and only if $\langle x, y \rangle \in \mathbf{R}_{\operatorname{succ}(\operatorname{succ} A)}$.

(52) If X is transitive and $\mathbf{R}_A \cap \mathbf{T}(X) = \mathbf{R}_{\operatorname{succ} A} \cap \mathbf{T}(X)$, then $\mathbf{T}(X) \subseteq \mathbf{R}_A$.

(53) If X is transitive, then there exists A such that $\mathbf{T}(X) \subseteq \mathbf{R}_A$.

(54) If X is transitive, then $\bigcup X \subseteq X$.

(55) If X is transitive and Y is transitive, then $X \cup Y$ is transitive.

(56) If X is transitive and Y is transitive, then $X \cap Y$ is transitive.

In the sequel k, n denote natural numbers. Let us consider X. The functor $X^{*\in}$ yielding a set, is defined by:

$x \in X^{*\in}$ if and only if there exist f, n, Y such that $x \in Y$ and $Y = f(n)$ and $\mathrm{dom}\, f = \mathbb{N}$ and $f(0) = X$ and for all k, y such that $y = f(k)$ holds $f(k+1) = \bigcup[y]$.

Next we state a number of propositions:

(57) $Z = X^{*\in}$ if and only if for every x holds $x \in Z$ if and only if there exist f, n, Y such that $x \in Y$ and $Y = f(n)$ and $\mathrm{dom}\, f = \mathbb{N}$ and $f(0) = X$ and for all k, y such that $y = f(k)$ holds $f(k+1) = \bigcup[y]$.

(58) $X^{*\in}$ is transitive.

(59) $X \subseteq X^{*\in}$.

(60) If $X \subseteq Y$ and Y is transitive, then $X^{*\in} \subseteq Y$.

(61) If for every Z such that $X \subseteq Z$ and Z is transitive holds $Y \subseteq Z$ and $X \subseteq Y$ and Y is transitive, then $X^{*\in} = Y$.

(62) If X is transitive, then $X^{*\in} = X$.

(63) $\emptyset^{*\in} = \emptyset$.

(64) $A^{*\in} = A$.

(65) If $X \subseteq Y$, then $X^{*\in} \subseteq Y^{*\in}$.

(66) $(X^{*\in})^{*\in} = X^{*\in}$.

(67) $(X \cup Y)^{*\in} = X^{*\in} \cup Y^{*\in}$.

(68) $(X \cap Y)^{*\in} \subseteq X^{*\in} \cap Y^{*\in}$.

(69) There exists A such that $X \subseteq \mathbf{R}_A$.

Let us consider X. The functor $\mathrm{rk}(X)$ yielding an ordinal number, is defined by:

$X \subseteq \mathbf{R}_{\mathrm{rk}(X)}$ and for every B such that $X \subseteq \mathbf{R}_B$ holds $\mathrm{rk}(X) \subseteq B$.

We now state a number of propositions:

(70) $A = \mathrm{rk}(X)$ if and only if $X \subseteq \mathbf{R}_A$ and for every B such that $X \subseteq \mathbf{R}_B$ holds $A \subseteq B$.

(71) $\mathrm{rk}(2^X) = \mathrm{succ}\,\mathrm{rk}(X)$.

(72) $\mathrm{rk}(\mathbf{R}_A) = A$.

(73) $X \subseteq \mathbf{R}_A$ if and only if $\mathrm{rk}(X) \subseteq A$.

(74) $X \in \mathbf{R}_A$ if and only if $\mathrm{rk}(X) \in A$.

(75) If $X \subseteq Y$, then $\mathrm{rk}(X) \subseteq \mathrm{rk}(Y)$.

(76) If $X \in Y$, then $\mathrm{rk}(X) \in \mathrm{rk}(Y)$.

(77) $\mathrm{rk}(X) \subseteq A$ if and only if for every Y such that $Y \in X$ holds $\mathrm{rk}(Y) \in A$.

(78) $A \subseteq \mathrm{rk}(X)$ if and only if for every B such that $B \in A$ there exists Y such that $Y \in X$ and $B \subseteq \mathrm{rk}(Y)$.

(79) $\mathrm{rk}(X) = \mathbf{0}$ if and only if $X = \emptyset$.

(80) If $\mathrm{rk}(X) = \mathrm{succ}\, A$, then there exists Y such that $Y \in X$ and $\mathrm{rk}(Y) = A$.

(81) $\mathrm{rk}(A) = A$.

(82) $\mathrm{rk}(\mathbf{T}(X)) \neq \mathbf{0}$ and $\mathrm{rk}(\mathbf{T}(X))$ is a limit ordinal number.

References

[1] Grzegorz Bancerek. Cardinal arithmetics. *Formalized Mathematics*, 1(**3**):543–547, 1990.

[2] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[4] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[5] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.

[6] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(**2**):265–267, 1990.

[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[8] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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