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Universal Classes

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Summary. In the article we have shown that there exist universal classes, i.e. there are sets which are closed w.r.t. basic set theory operations.

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The articles [11], [8], [4], [7], [10], [9], [5], [2], [1], [6], and [3] provide the terminology and notation for this paper. For simplicity we adopt the following convention: m is a cardinal number, A, B, C are ordinal numbers, x, y are arbitrary, and X, Y, W are sets. One can prove the following propositions:

(1) If W is a Tarski-Class and $X \in W$, then $X \not\approx W$ and $\overline{\overline{X}} < \overline{\overline{W}}$.

(2) If W is a Tarski-Class and $X \subseteq W$ and $\overline{\overline{X}} < \overline{\overline{W}}$, then $X \in W$.

(3) If W is a Tarski-Class and $x \in W$ and $y \in W$, then $\{x\} \in W$ and $\{x, y\} \in W$.

(4) If W is a Tarski-Class and $x \in W$ and $y \in W$, then $\langle x, y \rangle \in W$.

(5) If W is a Tarski-Class and $X \in W$, then $\mathbf{T}(X) \subseteq W$.

The scheme TC deals with a unary predicate $\mathcal{P}$, and states that:
for every X holds $\mathcal{P}[\mathbf{T}(X)]$
provided the parameter fulfills the following condition:

- for every X such that X is a Tarski-Class holds $\mathcal{P}[X]$.

Next we state a number of propositions:

(6) If W is a Tarski-Class and $A \in W$, then $\operatorname{succ} A \in W$ and $A \subseteq W$.

(7) If $A \in \mathbf{T}(W)$, then $\operatorname{succ} A \in \mathbf{T}(W)$ and $A \subseteq \mathbf{T}(W)$.

(8) If W is a Tarski-Class and X is transitive and $X \in W$, then $X \subseteq W$.

(9) If X is transitive and $X \in \mathbf{T}(W)$, then $X \subseteq \mathbf{T}(W)$.

(10) If W is a Tarski-Class, then $\operatorname{On} W = \overline{\overline{W}}$.

(11) $\operatorname{On} \mathbf{T}(W) = \overline{\overline{\mathbf{T}(W)}}$.

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(12) If W is a Tarski-Class and $X \in W$, then $\overline{\overline{X}} \in W$.

(13) If $X \in \mathbf{T}(W)$, then $\overline{\overline{X}} \in \mathbf{T}(W)$.

(14) If W is a Tarski-Class and $x \in \mathrm{ord}(\overline{\overline{W}})$, then $x \in W$.

(15) If $x \in \mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})$, then $x \in \mathbf{T}(W)$.

(16) If W is a Tarski-Class and $m < \overline{\overline{W}}$, then $m \in W$.

(17) If $m < \overline{\overline{\mathbf{T}(W)}}$, then $m \in \mathbf{T}(W)$.

(18) If W is a Tarski-Class and $m \in W$, then $m \subseteq W$.

(19) If $m \in \mathbf{T}(W)$, then $m \subseteq \mathbf{T}(W)$.

(20) If W is a Tarski-Class, then $\mathrm{ord}(\overline{\overline{W}})$ is a limit ordinal number.

(21) If W is a Tarski-Class and $W \neq \emptyset$, then $\overline{\overline{W}} \neq \overline{\mathbf{0}}$ and $\mathrm{ord}(\overline{\overline{W}}) \neq \mathbf{0}$ and $\mathrm{ord}(\overline{\overline{W}})$ is a limit ordinal number.

(22) $\overline{\overline{\mathbf{T}(W)}} \neq \overline{\mathbf{0}}$ and $\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}}) \neq \mathbf{0}$ and $\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})$ is a limit ordinal number.

In the sequel L, L_1 are transfinite sequences. We now state a number of propositions:

(23) If W is a Tarski-Class but $X \in W$ and W is transitive or $X \in W$ and $X \subseteq W$ or $\overline{\overline{X}} < \overline{\overline{W}}$ and $X \subseteq W$, then $W^X \subseteq W$.

(24) If $X \in \mathbf{T}(W)$ and W is transitive or $X \in \mathbf{T}(W)$ and $X \subseteq \mathbf{T}(W)$ or $\overline{\overline{X}} < \overline{\overline{\mathbf{T}(W)}}$ and $X \subseteq \mathbf{T}(W)$, then $\mathbf{T}(W)^X \subseteq \mathbf{T}(W)$.

(25) If $\mathrm{dom}\, L$ is a limit ordinal number and for every A such that $A \in \mathrm{dom}\, L$ holds $L(A) = \mathbf{R}_A$, then $\mathbf{R}_{\mathrm{dom}\, L} = \bigcup L$.

(26) If W is a Tarski-Class and $A \in \mathrm{On}\, W$, then $\overline{\overline{\mathbf{R}_A}} < \overline{\overline{W}}$ and $\mathbf{R}_A \in W$.

(27) If $A \in \mathrm{On}\, \mathbf{T}(W)$, then $\overline{\overline{\mathbf{R}_A}} < \overline{\overline{\mathbf{T}(W)}}$ and $\mathbf{R}_A \in \mathbf{T}(W)$.

(28) If W is a Tarski-Class, then $\mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})} \subseteq W$.

(29) $\mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})} \subseteq \mathbf{T}(W)$.

(30) If W is a Tarski-Class and W is transitive and $X \in W$, then $\mathrm{rk}(X) \in W$.

(31) If W is a Tarski-Class and W is transitive, then $W \subseteq \mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}$.

(32) If W is a Tarski-Class and W is transitive, then $\mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})} = W$.

(33) If W is a Tarski-Class and $A \in \mathrm{On}\, W$, then $\overline{\overline{\mathbf{R}_A}} \leq \overline{\overline{W}}$.

(34) If $A \in \mathrm{On}\, \mathbf{T}(W)$, then $\overline{\overline{\mathbf{R}_A}} \leq \overline{\overline{\mathbf{T}(W)}}$.

(35) If W is a Tarski-Class, then $\overline{\overline{W}} = \overline{\overline{\mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}}}$.

(36) $\overline{\overline{\mathbf{T}(W)}} = \overline{\overline{\mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}}}$.

(37) If W is a Tarski-Class and $X \subseteq \mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}$, then $X \approx \mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}$ or $X \in \mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}$.

(38) If $X \subseteq \mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}$, then $X \approx \mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}$ or $X \in \mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}$.

(39) If W is a Tarski-Class, then $\mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}$ is a Tarski-Class.

(40) $\mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}$ is a Tarski-Class.

(41) If X is transitive and $A \in \mathrm{rk}(X)$, then there exists Y such that $Y \in X$ and $\mathrm{rk}(Y) = A$.

(42) If X is transitive, then $\overline{\overline{\mathrm{rk}(X)}} \leq \overline{\overline{X}}$.

(43) If W is a Tarski-Class and X is transitive and $X \in W$, then $X \in \mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}$.

(44) If X is transitive and $X \in \mathbf{T}(W)$, then $X \in \mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}$.

(45) If W is transitive, then $\mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}$ is Tarski-Class of W.

(46) If W is transitive, then $\mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})} = \mathbf{T}(W)$.

A non-empty family of sets is called a universal class if:
it is transitive and it is a Tarski-Class.

In the sequel M denotes a non-empty family of sets. The following proposition is true

(47) For every M holds M is a universal class if and only if M is transitive and M is a Tarski-Class.

In the sequel U_1, U_2, U_3, $Universum$ will be universal classes. We now state several propositions:

(48) If $X \in Universum$, then $X \subseteq Universum$.

(49) If $X \in Universum$ and $Y \subseteq X$, then $Y \in Universum$.

(50) $\mathrm{On}\,Universum$ is an ordinal number.

(51) If X is transitive, then $\mathbf{T}(X)$ is a universal class.

(52) $\mathbf{T}(Universum)$ is a universal class.

Let us consider $Universum$. Then $\mathrm{On}\,Universum$ is an ordinal number. Then $\mathbf{T}(Universum)$ is a universal class.

Next we state a proposition

(53) $\mathbf{T}(A)$ is a universal class.

Let us consider A. Then $\mathbf{T}(A)$ is a universal class.

Next we state a number of propositions:

(54) $Universum = \mathbf{R}_{\mathrm{On}\,Universum}$.

(55) $\mathrm{On}\,Universum \neq \mathbf{0}$ and $\mathrm{On}\,Universum$ is a limit ordinal number.

(56) $U_1 \in U_2$ or $U_1 = U_2$ or $U_2 \in U_1$.

(57) $U_1 \subseteq U_2$ or $U_2 \in U_1$.

(58) $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

(59) If $U_1 \in U_2$ and $U_2 \in U_3$, then $U_1 \in U_3$.

(60) If $U_1 \subseteq U_2$ and $U_2 \in U_3$, then $U_1 \in U_3$.

(61) $U_1 \cup U_2$ is a universal class and $U_1 \cap U_2$ is a universal class.

(62) $\emptyset \in Universum$.

(63) If $x \in Universum$, then $\{x\} \in Universum$.

(64) If $x \in Universum$ and $y \in Universum$, then $\{x, y\} \in Universum$ and $\langle x, y\rangle \in Universum$.

(65) If $X \in Universum$, then $2^X \in Universum$ and $\bigcup X \in Universum$ and $\bigcap X \in Universum$.

(66) If $X \in Universum$ and $Y \in Universum$, then $X \cup Y \in Universum$ and $X \cap Y \in Universum$ and $X \setminus Y \in Universum$ and $X \dot{-} Y \in Universum$.

(67) If $X \in Universum$ and $Y \in Universum$, then $[\!: X, Y :\!] \in Universum$ and $Y^X \in Universum$.

In the sequel u, v are elements of $Universum$. Let us consider $Universum$, u. Then $\{u\}$ is an element of $Universum$. Then 2^u is an element of $Universum$. Then $\bigcup u$ is an element of $Universum$. Then $\bigcap u$ is an element of $Universum$. Let us consider v. Then $\{u, v\}$ is an element of $Universum$. Then $\langle u, v\rangle$ is an element of $Universum$. Then $u \cup v$ is an element of $Universum$. Then $u \cap v$ is an element of $Universum$. Then $u \setminus v$ is an element of $Universum$. Then $u \dot{-} v$ is an element of $Universum$. Then $[\!: u, v :\!]$ is an element of $Universum$. Then v^u is an element of $Universum$.

The universal class $\mathbf{U}_0$ is defined as follows:

$\mathbf{U}_0 = \mathbf{T}(\mathbf{0})$.

We now state four propositions:

(68) $\mathbf{U}_0 = \mathbf{T}(\mathbf{0})$.

(69) $\overline{\overline{\mathbf{R}_\omega}} = \overline{\overline{\omega}}$.

(70) $\mathbf{R}_\omega$ is a Tarski-Class.

(71) $\mathbf{U}_0 = \mathbf{R}_\omega$.

The universal class $\mathbf{U}_1$ is defined by:

$\mathbf{U}_1 = \mathbf{T}(\mathbf{U}_0)$.

The following proposition is true

(72) $\mathbf{U}_1 = \mathbf{T}(\mathbf{U}_0)$.

We now define three new constructions. A set of a finite rank is an element of $\mathbf{U}_0$.

A *Set* is an element of $\mathbf{U}_1$.

Let us consider A. The functor $\mathbf{U}_A$ is defined as follows:

there exists L such that $\mathbf{U}_A = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\mathbf{0}) = \mathbf{U}_0$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} A$ and $y = L(C)$ holds $L(\operatorname{succ} C) = \mathbf{T}([y])$ and for all C, L_1 such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathbf{T}(\bigcup L_1)$.

The following two propositions are true:

(73) For every element u of $\mathbf{U}_0$ holds u is a set of a finite rank.

(74) For every element u of $\mathbf{U}_1$ holds u is a *Set*.

Let u be a set of a finite rank. Then $\{u\}$ is a set of a finite rank. Then 2^u is a set of a finite rank. Then $\bigcup u$ is a set of a finite rank. Then $\bigcap u$ is a set of a finite rank. Let v be a set of a finite rank. Then $\{u, v\}$ is a set of a finite rank. Then $\langle u, v \rangle$ is a set of a finite rank. Then $u \cup v$ is a set of a finite rank. Then $u \cap v$ is a set of a finite rank. Then $u \setminus v$ is a set of a finite rank. Then $u \dot{-} v$ is a set of a finite rank. Then $[\!: u, v :\!]$ is a set of a finite rank. Then v^u is a set of a finite rank.

Let u be a *Set*. Then $\{u\}$ is a *Set*. Then 2^u is a *Set*. Then $\bigcup u$ is a *Set*. Then $\bigcap u$ is a *Set*. Let v be a *Set*. Then $\{u, v\}$ is a *Set*. Then $\langle u, v \rangle$ is a *Set*. Then $u \cup v$ is a *Set*. Then $u \cap v$ is a *Set*. Then $u \setminus v$ is a *Set*. Then $u \dot{-} v$ is a *Set*. Then $[\!: u, v :\!]$ is a *Set*. Then v^u is a *Set*.

Let us consider A. Then $\mathbf{U}_A$ is a universal class.

We now state several propositions:

(75) $\mathbf{U}_{\mathbf{0}} = \mathbf{U}_0$.

(76) $\mathbf{U}_{\mathrm{succ}\, A} = \mathbf{T}(\mathbf{U}_A)$.

(77) $\mathbf{U}_{\mathbf{1}} = \mathbf{U}_1$.

(78) If $A \neq \mathbf{0}$ and A is a limit ordinal number and $\mathrm{dom}\, L = A$ and for every B such that $B \in A$ holds $L(B) = \mathbf{U}_B$, then $\mathbf{U}_A = \mathbf{T}(\bigcup L)$.

(79) $\mathbf{U}_0 \subseteq Universum$ and $\mathbf{T}(\mathbf{0}) \subseteq Universum$ and $\mathbf{U}_{\mathbf{0}} \subseteq Universum$.

(80) $A \in B$ if and only if $\mathbf{U}_A \in \mathbf{U}_B$.

(81) If $\mathbf{U}_A = \mathbf{U}_B$, then $A = B$.

(82) $A \subseteq B$ if and only if $\mathbf{U}_A \subseteq \mathbf{U}_B$.

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