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The Complex Numbers

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Summary. We define the set $\mathbb{C}$ of complex numbers as the set of all ordered pairs $z = \langle a, b \rangle$ where a and b are real numbers and where addition and multiplication are defined. We define the real and imaginary parts of z and denote this by $a = \Re(z)$, $b = \Im(z)$. These definitions satisfy all the axioms for a field. $0_{\mathbb{C}} = 0 + 0i$ and $1_{\mathbb{C}} = 1 + 0i$ are identities for addition and multiplication respectively, and there are multiplicative inverses for each non zero element in $\mathbb{C}$. The difference and division of complex numbers are also defined. We do not interpret the set of all real numbers $\mathbb{R}$ as a subset of $\mathbb{C}$. From here on we do not abandon the ordered pair notation for complex numbers. For example: $i^2 = (0+1i)^2 = -1 + 0i \neq -1$. We conclude this article by introducing two operations on $\mathbb{C}$ which are not field operations. We define the absolute value of z denoted by $|z|$ and the conjugate of z denoted by z^*.

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The articles [1], [3], [2], and [4] provide the notation and terminology for this paper. In the sequel a, b, a_1, b_1, a_2, b_2 denote real numbers. The following two propositions are true:

(1) If $a \neq 0$, then $\frac{0}{a} = 0$.

(2) $a^{\mathbf{2}} + b^{\mathbf{2}} = 0$ if and only if $a = 0$ and $b = 0$.

The non-empty set $\mathbb{C}$ is defined as follows:

$\mathbb{C} = [\!: \mathbb{R}, \mathbb{R} :\!]$.

One can prove the following proposition

(3) $\mathbb{C} = [\!: \mathbb{R}, \mathbb{R} :\!]$.

In the sequel z, z_1, z_2, z_3, z_4 will denote elements of $\mathbb{C}$. We now define two new functors. Let us consider z. The functor $\Re(z)$ yielding a real number, is defined by:

$\Re(z) = z_{\mathbf{1}}$.

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The functor $\Im(z)$ yielding a real number, is defined as follows:
$\Im(z) = z_{\mathbf{2}}$.

We now state two propositions:

(4) $\Re(z) = z_{\mathbf{1}}$.

(5) $\Im(z) = z_{\mathbf{2}}$.

Let x, y be elements of $\mathbb{R}$. The functor $x + yi$ yields an element of $\mathbb{C}$ and is defined as follows:
$x + yi = \langle x, y \rangle$.

Next we state several propositions:

(6) For all elements x, y of $\mathbb{R}$ holds $x + yi = \langle x, y \rangle$.

(7) $\Re(a + bi) = a$ and $\Im(a + bi) = b$.

(8) $\Re(z) + \Im(z)i = z$.

(9) If $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$, then $z_1 = z_2$.

(10) If $a_1 + b_1 i = a_2 + b_2 i$, then $a_1 = a_2$ and $b_1 = b_2$.

Let us consider z_1, z_2. Let us note that one can characterize the predicate $z_1 = z_2$ by the following (equivalent) condition: $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$.

We now define three new functors. The element $0_{\mathbb{C}}$ of $\mathbb{C}$ is defined as follows:
$0_{\mathbb{C}} = 0 + 0i$.
The element $1_{\mathbb{C}}$ of $\mathbb{C}$ is defined by:
$1_{\mathbb{C}} = 1 + 0i$.
The element i of $\mathbb{C}$ is defined as follows:
$i = 0 + 1i$.

The following propositions are true:

(11) $0_{\mathbb{C}} = 0 + 0i$.

(12) $\Re(0_{\mathbb{C}}) = 0$ and $\Im(0_{\mathbb{C}}) = 0$.

(13) $z = 0_{\mathbb{C}}$ if and only if $\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}} = 0$.

(14) $1_{\mathbb{C}} = 1 + 0i$.

(15) $\Re(1_{\mathbb{C}}) = 1$ and $\Im(1_{\mathbb{C}}) = 0$.

(16) $i = 0 + 1i$.

(17) $\Re(i) = 0$ and $\Im(i) = 1$.

Let us consider z_1, z_2. The functor $z_1 + z_2$ yields an element of $\mathbb{C}$ and is defined as follows:
$z_1 + z_2 = \Re(z_1) + \Re(z_2) + \Im(z_1) + \Im(z_2)i$.

We now state several propositions:

(18) $z_1 + z_2 = \Re(z_1) + \Re(z_2) + \Im(z_1) + \Im(z_2)i$.

(19) $\Re(z_1 + z_2) = \Re(z_1) + \Re(z_2)$ and $\Im(z_1 + z_2) = \Im(z_1) + \Im(z_2)$.

(20) $z_1 + z_2 = z_2 + z_1$.

(21) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

(22) $0_{\mathbb{C}} + z = z$ and $z + 0_{\mathbb{C}} = z$.

Let us consider z_1, z_2. The functor $z_1 \cdot z_2$ yielding an element of $\mathbb{C}$, is defined as follows:

$z_1 \cdot z_2 = \Re(z_1) \cdot \Re(z_2) - \Im(z_1) \cdot \Im(z_2) + \Re(z_1) \cdot \Im(z_2) + \Re(z_2) \cdot \Im(z_1) i$.

Next we state a number of propositions:

(23) $z_1 \cdot z_2 = \Re(z_1) \cdot \Re(z_2) - \Im(z_1) \cdot \Im(z_2) + \Re(z_1) \cdot \Im(z_2) + \Re(z_2) \cdot \Im(z_1) i$.

(24) $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2) - \Im(z_1) \cdot \Im(z_2)$ and $\Im(z_1 \cdot z_2) = \Re(z_1) \cdot \Im(z_2) + \Re(z_2) \cdot \Im(z_1)$.

(25) $z_1 \cdot z_2 = z_2 \cdot z_1$.

(26) $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$.

(27) $z \cdot (z_1 + z_2) = z \cdot z_1 + z \cdot z_2$ and $(z_1 + z_2) \cdot z = z_1 \cdot z + z_2 \cdot z$.

(28) $0_{\mathbb{C}} \cdot z = 0_{\mathbb{C}}$ and $z \cdot 0_{\mathbb{C}} = 0_{\mathbb{C}}$.

(29) $1_{\mathbb{C}} \cdot z = z$ and $z \cdot 1_{\mathbb{C}} = z$.

(30) If $\Im(z_1) = 0$ and $\Im(z_2) = 0$, then $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2)$ and $\Im(z_1 \cdot z_2) = 0$.

(31) If $\Re(z_1) = 0$ and $\Re(z_2) = 0$, then $\Re(z_1 \cdot z_2) = -\Im(z_1) \cdot \Im(z_2)$ and $\Im(z_1 \cdot z_2) = 0$.

(32) $\Re(z \cdot z) = \Re(z)^{\mathbf{2}} - \Im(z)^{\mathbf{2}}$ and $\Im(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im(z))$.

Let us consider z. The functor $-z$ yielding an element of $\mathbb{C}$, is defined by:

$-z = -\Re(z) + -\Im(z) i$.

One can prove the following propositions:

(33) $-z = -\Re(z) + -\Im(z) i$.

(34) $\Re(-z) = -\Re(z)$ and $\Im(-z) = -\Im(z)$.

(35) $-0_{\mathbb{C}} = 0_{\mathbb{C}}$.

(36) If $-z = 0_{\mathbb{C}}$, then $z = 0_{\mathbb{C}}$.

(37) $i \cdot i = -1_{\mathbb{C}}$.

(38) $z + (-z) = 0_{\mathbb{C}}$ and $(-z) + z = 0_{\mathbb{C}}$.

(39) If $z_1 + z_2 = 0_{\mathbb{C}}$, then $z_2 = -z_1$ and $z_1 = -z_2$.

(40) $-(-z) = z$.

(41) If $-z_1 = -z_2$, then $z_1 = z_2$.

(42) If $z_1 + z = z_2 + z$ or $z_1 + z = z + z_2$, then $z_1 = z_2$.

(43) $-(z_1 + z_2) = (-z_1) + (-z_2)$.

(44) $(-z_1) \cdot z_2 = -z_1 \cdot z_2$ and $z_1 \cdot (-z_2) = -z_1 \cdot z_2$.

(45) $(-z_1) \cdot (-z_2) = z_1 \cdot z_2$.

(46) $-z = (-1_{\mathbb{C}}) \cdot z$.

Let us consider z_1, z_2. The functor $z_1 - z_2$ yields an element of $\mathbb{C}$ and is defined by:

$z_1 - z_2 = \Re(z_1) - \Re(z_2) + \Im(z_1) - \Im(z_2) i$.

We now state a number of propositions:

(47) $z_1 - z_2 = \Re(z_1) - \Re(z_2) + \Im(z_1) - \Im(z_2) i$.

(48) $\Re(z_1 - z_2) = \Re(z_1) - \Re(z_2)$ and $\Im(z_1 - z_2) = \Im(z_1) - \Im(z_2)$.

(49) $z_1 - z_2 = z_1 + (-z_2)$.

(50) If $z_1 - z_2 = 0_{\mathbb{C}}$, then $z_1 = z_2$.

(51) $z - z = 0_{\mathbb{C}}$.

(52) $z - 0_{\mathbb{C}} = z$.

(53) $0_{\mathbb{C}} - z = -z$.

(54) $z_1 - (-z_2) = z_1 + z_2$.

(55) $-(z_1 - z_2) = (-z_1) + z_2$.

(56) $-(z_1 - z_2) = z_2 - z_1$.

(57) $z_1 + (z_2 - z_3) = (z_1 + z_2) - z_3$.

(58) $z_1 - (z_2 - z_3) = (z_1 - z_2) + z_3$.

(59) $(z_1 - z_2) - z_3 = z_1 - (z_2 + z_3)$.

(60) $z_1 = (z_1 + z) - z$.

(61) $z_1 = (z_1 - z) + z$.

(62) $z \cdot (z_1 - z_2) = z \cdot z_1 - z \cdot z_2$ and $(z_1 - z_2) \cdot z = z_1 \cdot z - z_2 \cdot z$.

Let us consider z. The functor z^{-1} yields an element of $\mathbb{C}$ and is defined by: $z^{-1} = \frac{\Re(z)}{\Re(z)^{\mathbf{2}}+\Im(z)^{\mathbf{2}}} + \frac{-\Im(z)}{\Re(z)^{\mathbf{2}}+\Im(z)^{\mathbf{2}}} i$.

Next we state a number of propositions:

(63) $z^{-1} = \frac{\Re(z)}{\Re(z)^{\mathbf{2}}+\Im(z)^{\mathbf{2}}} + \frac{-\Im(z)}{\Re(z)^{\mathbf{2}}+\Im(z)^{\mathbf{2}}} i$.

(64) $\Re(z^{-1}) = \frac{\Re(z)}{\Re(z)^{\mathbf{2}}+\Im(z)^{\mathbf{2}}}$ and $\Im(z^{-1}) = \frac{-\Im(z)}{\Re(z)^{\mathbf{2}}+\Im(z)^{\mathbf{2}}}$.

(65) If $z \neq 0_{\mathbb{C}}$, then $z \cdot z^{-1} = 1_{\mathbb{C}}$ and $z^{-1} \cdot z = 1_{\mathbb{C}}$.

(66) If $z_1 \cdot z_2 = 0_{\mathbb{C}}$, then $z_1 = 0_{\mathbb{C}}$ or $z_2 = 0_{\mathbb{C}}$.

(67) If $z \neq 0_{\mathbb{C}}$, then $z^{-1} \neq 0_{\mathbb{C}}$.

(68) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$ and ${z_1}^{-1} = {z_2}^{-1}$, then $z_1 = z_2$.

(69) If $z_2 \neq 0_{\mathbb{C}}$ but $z_1 \cdot z_2 = 1_{\mathbb{C}}$ or $z_2 \cdot z_1 = 1_{\mathbb{C}}$, then $z_1 = {z_2}^{-1}$.

(70) If $z_2 \neq 0_{\mathbb{C}}$ but $z_1 \cdot z_2 = z_3$ or $z_2 \cdot z_1 = z_3$, then $z_1 = z_3 \cdot {z_2}^{-1}$ and $z_1 = {z_2}^{-1} \cdot z_3$.

(71) ${1_{\mathbb{C}}}^{-1} = 1_{\mathbb{C}}$.

(72) $i^{-1} = -i$.

(73) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then $(z_1 \cdot z_2)^{-1} = {z_1}^{-1} \cdot {z_2}^{-1}$.

(74) If $z \neq 0_{\mathbb{C}}$, then $(z^{-1})^{-1} = z$.

(75) If $z \neq 0_{\mathbb{C}}$, then $(-z)^{-1} = -z^{-1}$.

(76) If $z \neq 0_{\mathbb{C}}$ but $z_1 \cdot z = z_2 \cdot z$ or $z_1 \cdot z = z \cdot z_2$, then $z_1 = z_2$.

(77) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then ${z_1}^{-1} + {z_2}^{-1} = (z_1 + z_2) \cdot (z_1 \cdot z_2)^{-1}$.

(78) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then ${z_1}^{-1} - {z_2}^{-1} = (z_2 - z_1) \cdot (z_1 \cdot z_2)^{-1}$.

(79) If $\Re(z) \neq 0$ and $\Im(z) = 0$, then $\Re(z^{-1}) = \Re(z)^{-1}$ and $\Im(z^{-1}) = 0$.

(80) If $\Re(z) = 0$ and $\Im(z) \neq 0$, then $\Re(z^{-1}) = 0$ and $\Im(z^{-1}) = -\Im(z)^{-1}$.

Let us consider z_1, z_2. The functor $\frac{z_1}{z_2}$ yields an element of $\mathbb{C}$ and is defined by:

$\frac{z_1}{z_2} = \frac{\Re(z_1)\cdot\Re(z_2)+\Im(z_1)\cdot\Im(z_2)}{\Re(z_2)^{\mathbf{2}}+\Im(z_2)^{\mathbf{2}}} + \frac{\Re(z_2)\cdot\Im(z_1)-\Re(z_1)\cdot\Im(z_2)}{\Re(z_2)^{\mathbf{2}}+\Im(z_2)^{\mathbf{2}}} i.$

Next we state a number of propositions:

(81) $\frac{z_1}{z_2} = \frac{\Re(z_1)\cdot\Re(z_2)+\Im(z_1)\cdot\Im(z_2)}{\Re(z_2)^{\mathbf{2}}+\Im(z_2)^{\mathbf{2}}} + \frac{\Re(z_2)\cdot\Im(z_1)-\Re(z_1)\cdot\Im(z_2)}{\Re(z_2)^{\mathbf{2}}+\Im(z_2)^{\mathbf{2}}} i.$

(82) $\Re(\frac{z_1}{z_2}) = \frac{\Re(z_1)\cdot\Re(z_2)+\Im(z_1)\cdot\Im(z_2)}{\Re(z_2)^{\mathbf{2}}+\Im(z_2)^{\mathbf{2}}}$ and $\Im(\frac{z_1}{z_2}) = \frac{\Re(z_2)\cdot\Im(z_1)-\Re(z_1)\cdot\Im(z_2)}{\Re(z_2)^{\mathbf{2}}+\Im(z_2)^{\mathbf{2}}}$.

(83) If $z_2 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2} = z_1 \cdot {z_2}^{-1}$.

(84) If $z \neq 0_{\mathbb{C}}$, then $z^{-1} = \frac{1_{\mathbb{C}}}{z}$.

(85) $\frac{z}{1_{\mathbb{C}}} = z$.

(86) If $z \neq 0_{\mathbb{C}}$, then $\frac{z}{z} = 1_{\mathbb{C}}$.

(87) If $z \neq 0_{\mathbb{C}}$, then $\frac{0_{\mathbb{C}}}{z} = 0_{\mathbb{C}}$.

(88) If $z_2 \neq 0_{\mathbb{C}}$ and $\frac{z_1}{z_2} = 0_{\mathbb{C}}$, then $z_1 = 0_{\mathbb{C}}$.

(89) If $z_2 \neq 0_{\mathbb{C}}$ and $z_4 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2} \cdot \frac{z_3}{z_4} = \frac{z_1\cdot z_3}{z_2\cdot z_4}$.

(90) If $z_2 \neq 0_{\mathbb{C}}$, then $z \cdot \frac{z_1}{z_2} = \frac{z\cdot z_1}{z_2}$.

(91) If $z_2 \neq 0_{\mathbb{C}}$ and $\frac{z_1}{z_2} = 1_{\mathbb{C}}$, then $z_1 = z_2$.

(92) If $z \neq 0_{\mathbb{C}}$, then $z_1 = \frac{z_1\cdot z}{z}$.

(93) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2}^{-1} = \frac{z_2}{z_1}$.

(94) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then $\frac{{z_1}^{-1}}{{z_2}^{-1}} = \frac{z_2}{z_1}$.

(95) If $z_2 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{{z_2}^{-1}} = z_1 \cdot z_2$.

(96) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then $\frac{{z_1}^{-1}}{z_2} = (z_1 \cdot z_2)^{-1}$.

(97) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then ${z_1}^{-1} \cdot \frac{z}{z_2} = \frac{z}{z_1\cdot z_2}$.

(98) If $z \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2} = \frac{z_1\cdot z}{z_2\cdot z}$ and $\frac{z_1}{z_2} = \frac{z\cdot z_1}{z\cdot z_2}$.

(99) If $z_2 \neq 0_{\mathbb{C}}$ and $z_3 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2\cdot z_3} = \frac{\frac{z_1}{z_2}}{z_3}$.

(100) If $z_2 \neq 0_{\mathbb{C}}$ and $z_3 \neq 0_{\mathbb{C}}$, then $\frac{z_1\cdot z_3}{z_2} = \frac{z_1}{\frac{z_2}{z_3}}$.

(101) If $z_2 \neq 0_{\mathbb{C}}$ and $z_3 \neq 0_{\mathbb{C}}$ and $z_4 \neq 0_{\mathbb{C}}$, then $\frac{\frac{z_1}{z_2}}{\frac{z_3}{z_4}} = \frac{z_1\cdot z_4}{z_2\cdot z_3}$.

(102) If $z_2 \neq 0_{\mathbb{C}}$ and $z_4 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2} + \frac{z_3}{z_4} = \frac{z_1\cdot z_4+z_3\cdot z_2}{z_2\cdot z_4}$.

(103) If $z \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z} + \frac{z_2}{z} = \frac{z_1+z_2}{z}$.

(104) If $z_2 \neq 0_{\mathbb{C}}$, then $-\frac{z_1}{z_2} = \frac{-z_1}{z_2}$ and $-\frac{z_1}{z_2} = \frac{z_1}{-z_2}$.

(105) If $z_2 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2} = \frac{-z_1}{-z_2}$.

(106) If $z_2 \neq 0_{\mathbb{C}}$ and $z_4 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2} - \frac{z_3}{z_4} = \frac{z_1\cdot z_4-z_3\cdot z_2}{z_2\cdot z_4}$.

(107) If $z \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z} - \frac{z_2}{z} = \frac{z_1-z_2}{z}$.

(108) If $z_2 \neq 0_{\mathbb{C}}$ but $z_1 \cdot z_2 = z_3$ or $z_2 \cdot z_1 = z_3$, then $z_1 = \frac{z_3}{z_2}$.

(109) If $\Im(z_1) = 0$ and $\Im(z_2) = 0$ and $\Re(z_2) \neq 0$, then $\Re(\frac{z_1}{z_2}) = \frac{\Re(z_1)}{\Re(z_2)}$ and $\Im(\frac{z_1}{z_2}) = 0$.

(110) If $\Re(z_1) = 0$ and $\Re(z_2) = 0$ and $\Im(z_2) \neq 0$, then $\Re(\frac{z_1}{z_2}) = \frac{\Im(z_1)}{\Im(z_2)}$ and $\Im(\frac{z_1}{z_2}) = 0$.

Let us consider z. The functor z^* yielding an element of $\mathbb{C}$, is defined as follows:

$z^* = \Re(z) + -\Im(z)i$.

The following propositions are true:

(111) $z^* = \Re(z) + -\Im(z)i$.

(112) $\Re(z^*) = \Re(z)$ and $\Im(z^*) = -\Im(z)$.

(113) $0_{\mathbb{C}}{}^* = 0_{\mathbb{C}}$.

(114) If $z^* = 0_{\mathbb{C}}$, then $z = 0_{\mathbb{C}}$.

(115) $1_{\mathbb{C}}{}^* = 1_{\mathbb{C}}$.

(116) $i^* = -i$.

(117) $z^{**} = z$.

(118) $(z_1 + z_2)^* = z_1{}^* + z_2{}^*$.

(119) $(-z)^* = -z^*$.

(120) $(z_1 - z_2)^* = z_1{}^* - z_2{}^*$.

(121) $(z_1 \cdot z_2)^* = z_1{}^* \cdot z_2{}^*$.

(122) If $z \neq 0_{\mathbb{C}}$, then $(z^{-1})^* = z^{*-1}$.

(123) If $z_2 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2}{}^* = \frac{z_1{}^*}{z_2{}^*}$.

(124) If $\Im(z) = 0$, then $z^* = z$.

(125) If $\Re(z) = 0$, then $z^* = -z$.

(126) $\Re(z \cdot z^*) = \Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}$ and $\Im(z \cdot z^*) = 0$.

(127) $\Re(z + z^*) = 2 \cdot \Re(z)$ and $\Im(z + z^*) = 0$.

(128) $\Re(z - z^*) = 0$ and $\Im(z - z^*) = 2 \cdot \Im(z)$.

Let us consider z. The functor $|z|$ yielding a real number, is defined as follows:

$|z| = \sqrt{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}}$.

One can prove the following propositions:

(129) $|z| = \sqrt{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}}$.

(130) $|0_{\mathbb{C}}| = 0$.

(131) If $|z| = 0$, then $z = 0_{\mathbb{C}}$.

(132) $0 \leq |z|$.

(133) $z \neq 0_{\mathbb{C}}$ if and only if $0 < |z|$.

(134) $|1_{\mathbb{C}}| = 1$.

(135) $|i| = 1$.

(136) If $\Im(z) = 0$, then $|z| = |\Re(z)|$.

(137) If $\Re(z) = 0$, then $|z| = |\Im(z)|$.

(138) $|-z| = |z|$.

(139) $|z^*| = |z|$.

(140) $\Re(z) \leq |z|$.

(141) $\Im(z) \leq |z|$.

(142) $|z_1 + z_2| \leq |z_1| + |z_2|$.

(143) $|z_1 - z_2| \leq |z_1| + |z_2|$.

(144) $|z_1| - |z_2| \leq |z_1 + z_2|$.

(145) $|z_1| - |z_2| \leq |z_1 - z_2|$.

(146) $|z_1 - z_2| = |z_2 - z_1|$.

(147) $|z_1 - z_2| = 0$ if and only if $z_1 = z_2$.

(148) $z_1 \neq z_2$ if and only if $0 < |z_1 - z_2|$.

(149) $|z_1 - z_2| \leq |z_1 - z| + |z - z_2|$.

(150) $||z_1| - |z_2|| \leq |z_1 - z_2|$.

(151) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

(152) If $z \neq 0_{\mathbb{C}}$, then $|z^{-1}| = |z|^{-1}$.

(153) If $z_2 \neq 0_{\mathbb{C}}$, then $\frac{|z_1|}{|z_2|} = |\frac{z_1}{z_2}|$.

(154) $|z \cdot z| = \Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}$.

(155) $|z \cdot z| = |z \cdot z^*|$.

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