FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Function Domains and Frænkel Operator

Andrzej Trybulec[1]
Warsaw University, Białystok

Summary. We deal with a non–empty set of functions and a non–empty set of functions from a set A to a non–empty set B. In the case when B is a non–empty set, B^A is redefined. It yields a non–empty set of functions from A to B. An element of such a set is redefined as a function from A to B. Some theorems concerning these concepts are proved, as well as a number of schemes dealing with infinity and the Axiom of Choice. The article contains a number of schemes allowing for simple logical transformations related to terms constructed with the Frænkel Operator.

MML Identifier: FRAENKEL.

The articles [5], [4], [6], [1], [2], and [3] provide the notation and terminology for this paper. In the sequel A, B will be non-empty sets. We now state a proposition

(1) For arbitrary x holds $\{x\}$ is a non-empty set.

In the article we present several logical schemes. The scheme *Fraenkel5'* deals with a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{\mathcal{F}(v') : \mathcal{P}[v']\} \subseteq \{\mathcal{F}(u') : \mathcal{Q}[u']\}$

provided the parameters enjoy the following property:

- for every element v of $\mathcal{A}$ such that $\mathcal{P}[v]$ holds $\mathcal{Q}[v]$.

The scheme *Fraenkel5"* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{\mathcal{F}(u_1, v_1) : \mathcal{P}[u_1, v_1]\} \subseteq \{\mathcal{F}(u_2, v_2) : \mathcal{Q}[u_2, v_2]\}$

provided the following condition is fulfilled:

- for every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ such that $\mathcal{P}[u, v]$ holds $\mathcal{Q}[u, v]$.

The scheme *Fraenkel6'* deals with a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

[1]Supported by RPBP III.24 C1

ISSN 0777–4028

$\{\mathcal{F}(v_1) : \mathcal{P}[v_1]\} = \{\mathcal{F}(v_2) : \mathcal{Q}[v_2]\}$
provided the following requirement is fulfilled:

- for every element v of $\mathcal{A}$ holds $\mathcal{P}[v]$ if and only if $\mathcal{Q}[v]$.

The scheme *Fraenkel6"* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:
$\{\mathcal{F}(u_1, v_1) : \mathcal{P}[u_1, v_1]\} = \{\mathcal{F}(u_2, v_2) : \mathcal{Q}[u_2, v_2]\}$
provided the parameters fulfill the following requirement:

- for every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{P}[u, v]$ if and only if $\mathcal{Q}[u, v]$.

The scheme *FraenkelF'* concerns a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$, a unary functor $\mathcal{G}$, and a unary predicate $\mathcal{P}$, and states that:
$\{\mathcal{F}(v_1) : \mathcal{P}[v_1]\} = \{\mathcal{G}(v_2) : \mathcal{P}[v_2]\}$
provided the following requirement is met:

- for every element v of $\mathcal{A}$ holds $\mathcal{F}(v) = \mathcal{G}(v)$.

The scheme *FraenkelF'R* concerns a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$, a unary functor $\mathcal{G}$, and a unary predicate $\mathcal{P}$, and states that:
$\{\mathcal{F}(v_1) : \mathcal{P}[v_1]\} = \{\mathcal{G}(v_2) : \mathcal{P}[v_2]\}$
provided the parameters fulfill the following condition:

- for every element v of $\mathcal{A}$ such that $\mathcal{P}[v]$ holds $\mathcal{F}(v) = \mathcal{G}(v)$.

The scheme *FraenkelF"* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, and a binary predicate $\mathcal{P}$, and states that:
$\{\mathcal{F}(u_1, v_1) : \mathcal{P}[u_1, v_1]\} = \{\mathcal{G}(u_2, v_2) : \mathcal{P}[u_2, v_2]\}$
provided the parameters meet the following requirement:

- for every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{F}(u, v) = \mathcal{G}(u, v)$.

The scheme *FraenkelF6"C* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, and a binary predicate $\mathcal{P}$, and states that:
$\{\mathcal{F}(u_1, v_1) : \mathcal{P}[u_1, v_1]\} = \{\mathcal{F}(v_2, u_2) : \mathcal{P}[u_2, v_2]\}$
provided the following requirement is met:

- for every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{F}(u, v) = \mathcal{F}(v, u)$.

The scheme *FraenkelF6"* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:
$\{\mathcal{F}(u_1, v_1) : \mathcal{P}[u_1, v_1]\} = \{\mathcal{F}(v_2, u_2) : \mathcal{Q}[u_2, v_2]\}$
provided the parameters meet the following requirements:

- for every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{P}[u, v]$ if and only if $\mathcal{Q}[u, v]$,
- for every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{F}(u, v) = \mathcal{F}(v, u)$.

The following propositions are true:

(2) For all non-empty sets A, B and for every function F from A into B and for every set X and for every element x of A such that $x \in X$ holds $(F \upharpoonright X)(x) = F(x)$.

(3) For all non-empty sets A, B and for all functions F, G from A into B and for every set X such that $F \restriction X = G \restriction X$ for every element x of A such that $x \in X$ holds $F(x) = G(x)$.

(4) For every function f from A into B holds $f \in B^A$.

(5) For all sets A, B holds $B^A \subseteq 2^{[: A, B :]}$.

(6) For all sets X, Y such that $Y^X \neq \emptyset$ and $X \subseteq A$ and $Y \subseteq B$ for every element f of Y^X holds f is a partial function from A to B.

Now we present a number of schemes. The scheme *RelevantArgs* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a set $\mathcal{C}$, a function $\mathcal{D}$ from $\mathcal{A}$ into $\mathcal{B}$, a function $\mathcal{E}$ from $\mathcal{A}$ into $\mathcal{B}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{\mathcal{D}(u') : \mathcal{P}[u'] \wedge u' \in \mathcal{C}\} = \{\mathcal{E}(v') : \mathcal{Q}[v'] \wedge v' \in \mathcal{C}\}$

provided the following requirements are met:

- $\mathcal{D} \restriction \mathcal{C} = \mathcal{E} \restriction \mathcal{C}$,
- for every element u of $\mathcal{A}$ such that $u \in \mathcal{C}$ holds $\mathcal{P}[u]$ if and only if $\mathcal{Q}[u]$.

The scheme *Fr_Set0* deals with a non-empty set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\{xx : \mathcal{P}[xx]\} \subseteq \mathcal{A}$

for all values of the parameters.

The scheme *Gen1"* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

for every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ such that $\mathcal{P}[s,t]$ holds $\mathcal{Q}[\mathcal{F}(s,t)]$

provided the parameters meet the following requirement:

- for arbitrary s_t such that $s_t \in \{\mathcal{F}(s_1,t_1) : \mathcal{P}[s_1,t_1]\}$ holds $\mathcal{Q}[s_t]$.

The scheme *Gen1"A* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

for arbitrary s_t such that $s_t \in \{\mathcal{F}(s_1,t_1) : \mathcal{P}[s_1,t_1]\}$ holds $\mathcal{Q}[s_t]$

provided the following requirement is met:

- for every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ such that $\mathcal{P}[s,t]$ holds $\mathcal{Q}[\mathcal{F}(s,t)]$.

The scheme *Gen2"* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

$\{s_t : s_t \in \{\mathcal{F}(s_1,t_1) : \mathcal{P}[s_1,t_1]\} \wedge \mathcal{Q}[s_t]\} = \{\mathcal{F}(s_2,t_2) : \mathcal{P}[s_2,t_2] \wedge \mathcal{Q}[\mathcal{F}(s_2,t_2)]\}$

for all values of the parameters.

The scheme *Gen3'* concerns a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s) : s \in \{s_1 : \mathcal{Q}[s_1]\} \wedge \mathcal{P}[s]\} = \{\mathcal{F}(s_2) : \mathcal{Q}[s_2] \wedge \mathcal{P}[s_2]\}$

for all values of the parameters.

The scheme *Gen3"* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(s,t) : s \in \{s_1 : \mathcal{Q}[s_1]\} \wedge \mathcal{P}[s,t]\} = \{\mathcal{F}(s_2,t_2) : \mathcal{Q}[s_2] \wedge \mathcal{P}[s_2,t_2]\}$

for all values of the parameters.

The scheme *Gen4"* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s,t) : \mathcal{P}[s,t]\} \subseteq \{\mathcal{F}(s_1,t_1) : \mathcal{Q}[s_1,t_1]\}$

provided the following condition is satisfied:

- for every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ such that $\mathcal{P}[s,t]$ there exists an element s' of $\mathcal{A}$ such that $\mathcal{Q}[s',t]$ and $\mathcal{F}(s,t) = \mathcal{F}(s',t)$.

The scheme *FrSet_1* concerns a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(y) : \mathcal{F}(y) \in \mathcal{B} \wedge \mathcal{P}[y]\} \subseteq \mathcal{B}$

for all values of the parameters.

The scheme *FrSet_2* deals with a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(y) : \mathcal{P}[y] \wedge \mathcal{F}(y) \notin \mathcal{B}\}$ misses $\mathcal{B}$

for all values of the parameters.

The scheme *FrEqua1* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, an element $\mathcal{C}$ of $\mathcal{B}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s,t) : \mathcal{Q}[s,t]\} = \{\mathcal{F}(s',\mathcal{C}) : \mathcal{P}[s',\mathcal{C}]\}$

provided the parameters meet the following requirement:

- for every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ holds $\mathcal{Q}[s,t]$ if and only if $t = \mathcal{C}$ and $\mathcal{P}[s,t]$.

The scheme *FrEqua2* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, an element $\mathcal{C}$ of $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(s,t) : t = \mathcal{C} \wedge \mathcal{P}[s,t]\} = \{\mathcal{F}(s',\mathcal{C}) : \mathcal{P}[s',\mathcal{C}]\}$

for all values of the parameters.

A non-empty set is said to be a non-empty set of functions if:

for every element x of it holds x is a function.

Next we state two propositions:

(7) A is a non-empty set of functions if and only if for every element x of A holds x is a function.

(8) For every function f holds $\{f\}$ is a non-empty set of functions.

Let A be a set, and let B be a non-empty set. A non-empty set of functions is called a non-empty set of functions from A to B if:

for every element x of it holds x is a function from A into B.

Next we state three propositions:

(9) For every set A and for every non-empty set B and for every non-empty set C of functions holds C is a non-empty set of functions from A to B if and only if for every element x of C holds x is a function from A into B.

(10) For every function f from A into B holds $\{f\}$ is a non-empty set of functions from A to B.

(11) For every set A and for every non-empty set B holds B^A is a non-empty set of functions from A to B.

Let A be a set, and let B be a non-empty set. Then B^A is a non-empty set of functions from A to B. Let F be a non-empty set of functions from A to B. We see that it makes sense to consider the following mode for restricted scopes of arguments. Then all the objests of the mode element of F are a function from A into B.

In the sequel *phi* will be an element of B^A. The following propositions are true:

(12) For every function f from A into B holds f is an element of B^A.

(13) For every element f of B^A holds $\operatorname{dom} f = A$ and $\operatorname{rng} f \subseteq B$.

(14) For all sets X, Y such that $Y^X \neq \emptyset$ and $X \subseteq A$ and $Y \subseteq B$ for every element f of Y^X there exists an element *phi* of B^A such that $phi \restriction X = f$.

(15) For every set X and for every *phi* holds $phi \restriction X = phi \restriction (A \cap X)$.

Now we present four schemes. The scheme *FraenkelFin* deals with a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ and states that:

$\{\mathcal{F}(w) : w \in \mathcal{B}\}$ is finite

provided the parameters meet the following requirement:

- $\mathcal{B}$ is finite.

The scheme *CartFin* deals with a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, a set $\mathcal{C}$, and a binary functor $\mathcal{F}$ and states that:

$\{\mathcal{F}(u', v') : u' \in \mathcal{B} \wedge v' \in \mathcal{C}\}$ is finite

provided the parameters fulfill the following requirements:

- $\mathcal{B}$ is finite,
- $\mathcal{C}$ is finite.

The scheme *Finiteness* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\operatorname{Fin} \mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

for every element x of $\mathcal{A}$ such that $x \in \mathcal{B}$ there exists an element y of $\mathcal{A}$ such that $y \in \mathcal{B}$ and $\mathcal{P}[y, x]$ and for every element z of $\mathcal{A}$ such that $z \in \mathcal{B}$ and $\mathcal{P}[z, y]$ holds $\mathcal{P}[y, z]$

provided the following requirements are fulfilled:

- for every element x of $\mathcal{A}$ holds $\mathcal{P}[x, x]$,
- for all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

The scheme *Fin_Im* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\operatorname{Fin} \mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists an element c_1 of $\operatorname{Fin} \mathcal{A}$ such that for every element t of $\mathcal{A}$ holds $t \in c_1$ if and only if there exists an element t' of $\mathcal{B}$ such that $t' \in \mathcal{C}$ and $t = \mathcal{F}(t')$ and $\mathcal{P}[t, t']$

for all values of the parameters.

The following proposition is true

(16) For all sets A, B such that A is finite and B is finite holds B^A is finite.

Now we present three schemes. The scheme *ImFin* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a set $\mathcal{C}$, a set $\mathcal{D}$, and a unary functor $\mathcal{F}$ and states that:

$\{\mathcal{F}(phi') : phi' \circ \mathcal{C} \subseteq \mathcal{D}\}$ is finite

provided the parameters fulfill the following conditions:

- $\mathcal{C}$ is finite,
- $\mathcal{D}$ is finite,
- for all elements phi, psi of $\mathcal{B}^{\mathcal{A}}$ such that $phi \restriction \mathcal{C} = psi \restriction \mathcal{C}$ holds $\mathcal{F}(phi) = \mathcal{F}(psi)$.

The scheme *FunctChoice* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\operatorname{Fin}\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function ff from $\mathcal{A}$ into $\mathcal{B}$ such that for every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ holds $\mathcal{P}[t, ff(t)]$

provided the parameters fulfill the following condition:

- for every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ there exists an element ff of $\mathcal{B}$ such that $\mathcal{P}[t, ff]$.

The scheme *FuncsChoice* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\operatorname{Fin}\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists an element ff of $\mathcal{B}^{\mathcal{A}}$ such that for every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ holds $\mathcal{P}[t, ff(t)]$

provided the parameters meet the following requirement:

- for every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ there exists an element ff of $\mathcal{B}$ such that $\mathcal{P}[t, ff]$.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[4] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[6] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(**1**):187–190, 1990.

Received February 7, 1990