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The Modification of a Function by a Function and the Iteration of the Composition of a Function

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Summary. In the article we introduce some operation on functions. We define the natural ordering relation on functions. The fact that a function f is less than a function g we denote by $f \leq g$ and we define by graph$f \subseteq$ graph$f$. In the sequel we define the modifications of a function f by a function g denoted $f{+\cdot}g$ and the n-th iteration of the composition of a function f denoted by f^n. We prove some propositions related to the introduced notions.

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The papers [7], [1], [2], [3], [4], [5], and [6] provide the terminology and notation for this paper. For simplicity we adopt the following rules: a, b, x, x', y, y', z will be arbitrary, X, X', Y, Y', Z, Z' will be sets, D, D' will be non-empty sets, and f, g, h will be functions. We now state several propositions:

(1) If for every z such that $z \in Z$ there exist x, y such that $z = \langle x, y \rangle$, then there exist X, Y such that $Z \subseteq [\!: X, Y :\!]$.

(2) If $\mathrm{rng}\, f \cap \mathrm{dom}\, g = \emptyset$, then $g \cdot f = \square$.

(3) $g \cdot f = g \restriction \mathrm{rng}\, f \cdot f$.

(4) $\square = \emptyset \longmapsto a$.

(5) $\mathrm{graph}(\mathrm{id}_X) \subseteq \mathrm{graph}(\mathrm{id}_Y)$ if and only if $X \subseteq Y$.

(6) If $X \subseteq Y$, then $\mathrm{graph}(X \longmapsto a) \subseteq \mathrm{graph}(Y \longmapsto a)$.

(7) If $\mathrm{graph}(X \longmapsto a) \subseteq \mathrm{graph}(Y \longmapsto b)$, then $X \subseteq Y$.

(8) If $X \neq \emptyset$ and $\mathrm{graph}(X \longmapsto a) \subseteq \mathrm{graph}(Y \longmapsto b)$, then $a = b$.

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(9) If $x \in \mathrm{dom}\, f$, then $\mathrm{graph}(\{x\} \longmapsto f(x)) \subseteq \mathrm{graph}\, f$.

Let us consider f, g. The predicate $f \leq g$ is defined as follows:
$\mathrm{graph}\, f \subseteq \mathrm{graph}\, g$.

We now state a number of propositions:

(10) For all f, g holds $f \leq g$ if and only if $\mathrm{graph}\, f \subseteq \mathrm{graph}\, g$.

(11) $f \leq g$ if and only if $\mathrm{dom}\, f \subseteq \mathrm{dom}\, g$ and for every x such that $x \in \mathrm{dom}\, f$ holds $f(x) = g(x)$.

(12) If $f \leq g$, then $f \approx g$.

(13) If $f \leq g$, then $\mathrm{dom}\, f \subseteq \mathrm{dom}\, g$ and $\mathrm{rng}\, f \subseteq \mathrm{rng}\, g$.

(14) If $f \leq g$ and $\mathrm{dom}\, f = \mathrm{dom}\, g$, then $f = g$.

(15) $\square \leq f$.

(16) $f \leq f$.

(17) If $f \leq g$ and $g \leq h$, then $f \leq h$.

(18) $f \leq g$ and $g \leq f$ if and only if $f = g$.

(19) $\mathrm{id}_X \leq \mathrm{id}_Y$ if and only if $X \subseteq Y$.

(20) If $X \subseteq Y$, then $X \longmapsto a \leq Y \longmapsto a$.

(21) If $X \longmapsto a \leq Y \longmapsto b$, then $X \subseteq Y$.

(22) If $X \neq \emptyset$ and $X \longmapsto a \leq Y \longmapsto b$, then $a = b$.

(23) If $x \in \mathrm{dom}\, f$, then $\{x\} \longmapsto f(x) \leq f$.

(24) If $f \leq g$ and g is one-to-one, then f is one-to-one.

(25) If $f \leq g$, then $g \restriction \mathrm{dom}\, f = f$.

(26) If $f \leq g$ and g is one-to-one, then $\mathrm{rng}\, f \restriction g = f$.

(27) $f \restriction X \leq f$.

(28) If $X \subseteq Y$, then $f \restriction X \leq f \restriction Y$.

(29) If $X \subseteq Y$, then $X \restriction f \leq Y \restriction f$.

(30) $Y \restriction f \leq f$.

(31) $(Y \restriction f) \restriction X \leq f$.

(32) $f_{\restriction X \dot{\rightarrow} Y} \leq f$.

(33) If $f \leq g$, then $f \cdot h \leq g \cdot h$.

(34) If $f \leq g$, then $h \cdot f \leq h \cdot g$.

(35) For all functions f_1, f_2, g_1, g_2 such that $f_1 \leq g_1$ and $f_2 \leq g_2$ holds $f_1 \cdot f_2 \leq g_1 \cdot g_2$.

(36) If $f \leq g$, then $f \restriction X \leq g \restriction X$.

(37) If $f \leq g$, then $Y \restriction f \leq Y \restriction g$.

(38) If $f \leq g$, then $(Y \restriction f) \restriction X \leq (Y \restriction g) \restriction X$.

(39) If $f \leq g$, then $f_{\restriction X \dot{\rightarrow} Y} \leq g_{\restriction X \dot{\rightarrow} Y}$.

(40) If $f \leq h$ and $g \leq h$, then $f \approx g$.

Let us consider f, g. The functor $f +\!\cdot\, g$ yields a function and is defined by:

$\mathrm{dom}(f +\cdot g) = \mathrm{dom}\, f \cup \mathrm{dom}\, g$ and for every x such that $x \in \mathrm{dom}\, f \cup \mathrm{dom}\, g$ holds if $x \in \mathrm{dom}\, g$, then $(f +\cdot g)(x) = g(x)$ but if $x \notin \mathrm{dom}\, g$, then $(f +\cdot g)(x) = f(x)$.

We now state a number of propositions:

(41) Let f, g, h be functions. Then $h = f +\cdot g$ if and only if the following conditions are satisfied:

(i) $\mathrm{dom}\, h = \mathrm{dom}\, f \cup \mathrm{dom}\, g$,

(ii) for every x such that $x \in \mathrm{dom}\, f \cup \mathrm{dom}\, g$ holds if $x \in \mathrm{dom}\, g$, then $h(x) = g(x)$ but if $x \notin \mathrm{dom}\, g$, then $h(x) = f(x)$.

(42) If $x \in \mathrm{dom}(f +\cdot g)$ and $x \notin \mathrm{dom}\, g$, then $(f +\cdot g)(x) = f(x)$.

(43) $x \in \mathrm{dom}(f +\cdot g)$ if and only if $x \in \mathrm{dom}\, f$ or $x \in \mathrm{dom}\, g$.

(44) If $x \in \mathrm{dom}\, g$, then $(f +\cdot g)(x) = g(x)$.

(45) If $x \in \mathrm{dom}\, f \setminus \mathrm{dom}\, g$, then $(f +\cdot g)(x) = f(x)$.

(46) If $f \approx g$ and $x \in \mathrm{dom}\, f$, then $(f +\cdot g)(x) = f(x)$.

(47) If $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$ and $x \in \mathrm{dom}\, f$, then $(f +\cdot g)(x) = f(x)$.

(48) $\mathrm{rng}(f +\cdot g) \subseteq \mathrm{rng}\, f \cup \mathrm{rng}\, g$.

(49) $\mathrm{rng}\, g \subseteq \mathrm{rng}(f +\cdot g)$.

(50) If $\mathrm{dom}\, f \subseteq \mathrm{dom}\, g$, then $f +\cdot g = g$.

(51) If $\mathrm{dom}\, f = \mathrm{dom}\, g$, then $f +\cdot g = g$.

(52) $f +\cdot f = f$.

(53) $\square +\cdot f = f$.

(54) $f +\cdot \square = f$.

(55) $\mathrm{id}_X +\cdot \mathrm{id}_Y = \mathrm{id}_{X \cup Y}$.

(56) $(f +\cdot g) \restriction \mathrm{dom}\, g = g$.

(57) $\mathrm{graph}((f +\cdot g) \restriction (\mathrm{dom}\, f \setminus \mathrm{dom}\, g)) \subseteq \mathrm{graph}\, f$.

(58) $(f +\cdot g) \restriction (\mathrm{dom}\, f \setminus \mathrm{dom}\, g) \leq f$.

(59) $\mathrm{graph}\, g \subseteq \mathrm{graph}(f +\cdot g)$.

(60) $g \leq f +\cdot g$.

(61) If $f \approx g +\cdot h$, then $f \restriction (\mathrm{dom}\, f \setminus \mathrm{dom}\, h) \approx g$.

(62) If $f \approx g +\cdot h$, then $f \approx h$.

(63) $f \approx g$ if and only if $\mathrm{graph}\, f \subseteq \mathrm{graph}(f +\cdot g)$.

(64) $f \approx g$ if and only if $f \leq f +\cdot g$.

(65) $\mathrm{graph}(f +\cdot g) \subseteq \mathrm{graph}\, f \cup \mathrm{graph}\, g$.

(66) $f \approx g$ if and only if $\mathrm{graph}\, f \cup \mathrm{graph}\, g = \mathrm{graph}(f +\cdot g)$.

(67) If $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$, then $\mathrm{graph}\, f \cup \mathrm{graph}\, g = \mathrm{graph}(f +\cdot g)$.

(68) If $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$, then $\mathrm{graph}\, f \subseteq \mathrm{graph}(f +\cdot g)$.

(69) If $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$, then $f \leq f +\cdot g$.

(70) If $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$, then $(f +\cdot g) \restriction \mathrm{dom}\, f = f$.

(71) $f \approx g$ if and only if $f +\cdot g = g +\cdot f$.

(72) If $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$, then $f +\cdot g = g +\cdot f$.

(73) For all partial functions f, g from X to Y such that g is total holds $f +\cdot g = g$.

(74) For all functions f, g from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f +\cdot g = g$.

(75) For all functions f, g from X into X holds $f +\cdot g = g$.

(76) For all functions f, g from X into D holds $f +\cdot g = g$.

(77) For all partial functions f, g from X to Y holds $f +\cdot g$ is a partial function from X to Y.

Let us consider f. The functor $\curvearrowleft f$ yields a function and is defined by:

for every x holds $x \in \mathrm{dom}(\curvearrowleft f)$ if and only if there exist y, z such that $x = \langle z, y \rangle$ and $\langle y, z \rangle \in \mathrm{dom}\, f$ and for all y, z such that $\langle y, z \rangle \in \mathrm{dom}\, f$ holds $(\curvearrowleft f)(\langle z, y \rangle) = f(\langle y, z \rangle)$.

We now state a number of propositions:

(78) Let f, h be functions. Then $h = \curvearrowleft f$ if and only if for every z holds $z \in \mathrm{dom}\, h$ if and only if there exist x, y such that $z = \langle y, x \rangle$ and $\langle x, y \rangle \in \mathrm{dom}\, f$ and for all x, y such that $\langle x, y \rangle \in \mathrm{dom}\, f$ holds $h(\langle y, x \rangle) = f(\langle x, y \rangle)$.

(79) $\mathrm{rng}(\curvearrowleft f) \subseteq \mathrm{rng}\, f$.

(80) $\langle x, y \rangle \in \mathrm{dom}\, f$ if and only if $\langle y, x \rangle \in \mathrm{dom}(\curvearrowleft f)$.

(81) If $\langle y, x \rangle \in \mathrm{dom}(\curvearrowleft f)$, then $\curvearrowleft f(\langle y, x \rangle) = f(\langle x, y \rangle)$.

(82) There exist X, Y such that $\mathrm{dom}(\curvearrowleft f) \subseteq [\!: X, Y :\!]$.

(83) If $\mathrm{dom}\, f \subseteq [\!: X, Y :\!]$, then $\mathrm{dom}(\curvearrowleft f) \subseteq [\!: Y, X :\!]$.

(84) If $\mathrm{dom}\, f = [\!: X, Y :\!]$, then $\mathrm{dom}(\curvearrowleft f) = [\!: Y, X :\!]$.

(85) If $\mathrm{dom}\, f \subseteq [\!: X, Y :\!]$, then $\mathrm{rng}(\curvearrowleft f) = \mathrm{rng}\, f$.

(86) If $\mathrm{dom}\, f = [\!: X, Y :\!]$, then $\mathrm{rng}(\curvearrowleft f) = \mathrm{rng}\, f$.

(87) For every partial function f from $[\!: X, Y :\!]$ to Z holds $\curvearrowleft f$ is a partial function from $[\!: Y, X :\!]$ to Z.

(88) For every function f from $[\!: X, Y :\!]$ into Z such that $Z \neq \emptyset$ holds $\curvearrowleft f$ is a function from $[\!: Y, X :\!]$ into Z.

(89) For every function f from $[\!: X, Y :\!]$ into D holds $\curvearrowleft f$ is a function from $[\!: Y, X :\!]$ into D.

(90) $\mathrm{graph}(\curvearrowleft(\curvearrowleft f)) \subseteq \mathrm{graph}\, f$.

(91) If $\mathrm{dom}\, f \subseteq [\!: X, Y :\!]$, then $\curvearrowleft(\curvearrowleft f) = f$.

(92) If $\mathrm{dom}\, f = [\!: X, Y :\!]$, then $\curvearrowleft(\curvearrowleft f) = f$.

(93) For every partial function f from $[\!: X, Y :\!]$ to Z holds $\curvearrowleft(\curvearrowleft f) = f$.

(94) For every function f from $[\!: X, Y :\!]$ into Z such that $Z \neq \emptyset$ holds $\curvearrowleft(\curvearrowleft f) = f$.

(95) For every function f from $[\!: X, Y :\!]$ into D holds $\curvearrowleft(\curvearrowleft f) = f$.

Let us consider f, g. The functor $|\!: f, g :\!|$ yielding a function, is defined as follows:

(i) for every z holds $z \in \mathrm{dom}|\!: f, g :\!|$ if and only if there exist x, y, x', y' such that $z = \langle \langle x, x' \rangle, \langle y, y' \rangle \rangle$ and $\langle x, y \rangle \in \mathrm{dom}\, f$ and $\langle x', y' \rangle \in \mathrm{dom}\, g$,

(ii) for all x, y, x', y' such that $\langle x,y\rangle \in \mathrm{dom}\, f$ and $\langle x',y'\rangle \in \mathrm{dom}\, g$ holds $|{:}f,\, g{:}|(\langle\langle x,x'\rangle,\langle y,y'\rangle\rangle) = \langle f(\langle x,y\rangle), g(\langle x',y'\rangle)\rangle$.

The following propositions are true:

(96) Given f, g, h. Then $h = |{:}f,\, g{:}|$ if and only if the following conditions are satisfied:

(i) for every z holds $z \in \mathrm{dom}\, h$ if and only if there exist x, y, x', y' such that $z = \langle\langle x,x'\rangle,\langle y,y'\rangle\rangle$ and $\langle x,y\rangle \in \mathrm{dom}\, f$ and $\langle x',y'\rangle \in \mathrm{dom}\, g$,

(ii) for all x, y, x', y' such that $\langle x,y\rangle \in \mathrm{dom}\, f$ and $\langle x',y'\rangle \in \mathrm{dom}\, g$ holds $h(\langle\langle x,x'\rangle,\langle y,y'\rangle\rangle) = \langle f(\langle x,y\rangle), g(\langle x',y'\rangle)\rangle$.

(97) $\langle\langle x,x'\rangle,\langle y,y'\rangle\rangle \in \mathrm{dom}|{:}f,\, g{:}|$ if and only if $\langle x,y\rangle \in \mathrm{dom}\, f$ and $\langle x',y'\rangle \in \mathrm{dom}\, g$.

(98) If $\langle\langle x,x'\rangle,\langle y,y'\rangle\rangle \in \mathrm{dom}|{:}f,\, g{:}|$, then $|{:}f,\, g{:}|(\langle\langle x,x'\rangle,\langle y,y'\rangle\rangle) = \langle f(\langle x,y\rangle), g(\langle x',y'\rangle)\rangle$.

(99) $\mathrm{rng}|{:}f,\, g{:}| \subseteq [{:}\, \mathrm{rng}\, f,\, \mathrm{rng}\, g\, {:}]$.

(100) If $\mathrm{dom}\, f \subseteq [{:}\, X,\, Y\, {:}]$ and $\mathrm{dom}\, g \subseteq [{:}\, X',\, Y'\, {:}]$, then $\mathrm{dom}|{:}f,\, g{:}| \subseteq [{:}\, [{:}\, X,\, X'\, {:}],\, [{:}\, Y,\, Y'\, {:}]\, {:}]$.

(101) If $\mathrm{dom}\, f = [{:}\, X,\, Y\, {:}]$ and $\mathrm{dom}\, g = [{:}\, X',\, Y'\, {:}]$, then $\mathrm{dom}|{:}f,\, g{:}| = [{:}\, [{:}\, X,\, X'\, {:}],\, [{:}\, Y,\, Y'\, {:}]\, {:}]$.

(102) For every partial function f from $[{:}\, X,\, Y\, {:}]$ to Z and for every partial function g from $[{:}\, X',\, Y'\, {:}]$ to Z' holds $|{:}f,\, g{:}|$ is a partial function from $[{:}\, [{:}\, X,\, X'\, {:}],\, [{:}\, Y,\, Y'\, {:}]\, {:}]$ to $[{:}\, Z,\, Z'\, {:}]$.

(103) For every function f from $[{:}\, X,\, Y\, {:}]$ into Z and for every function g from $[{:}\, X',\, Y'\, {:}]$ into Z' such that $Z \neq \emptyset$ and $Z' \neq \emptyset$ holds $|{:}f,\, g{:}|$ is a function from $[{:}\, [{:}\, X,\, X'\, {:}],\, [{:}\, Y,\, Y'\, {:}]\, {:}]$ into $[{:}\, Z,\, Z'\, {:}]$.

(104) For every function f from $[{:}\, X,\, Y\, {:}]$ into D and for every function g from $[{:}\, X',\, Y'\, {:}]$ into D' holds $|{:}f,\, g{:}|$ is a function from $[{:}\, [{:}\, X,\, X'\, {:}],\, [{:}\, Y,\, Y'\, {:}]\, {:}]$ into $[{:}\, D,\, D'\, {:}]$.

Let f be a function, and let n be an element of $\mathbb{N}$. The functor f^n yields a function and is defined as follows:

there exists a function p from $\mathbb{N}$ into $(\mathrm{dom}\, f \cup \mathrm{rng}\, f) \dot{\rightarrow} (\mathrm{dom}\, f \cup \mathrm{rng}\, f)$ such that $f^n = p(n)$ and $p(0) = \mathrm{id}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f}$ and for every element k of $\mathbb{N}$ there exists a function g such that $g = p(k)$ and $p(k+1) = g \cdot f$.

One can prove the following proposition

(105) Let f be a function. Let n be an element of $\mathbb{N}$. Suppose $\mathrm{rng}\, f \subseteq \mathrm{dom}\, f$. Let h be a function. Then $h = f^n$ if and only if there exists a function p from $\mathbb{N}$ into $(\mathrm{dom}\, f \cup \mathrm{rng}\, f) \dot{\rightarrow} (\mathrm{dom}\, f \cup \mathrm{rng}\, f)$ such that $h = p(n)$ and $p(0) = \mathrm{id}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f}$ and for every element k of $\mathbb{N}$ there exists a function g such that $g = p(k)$ and $p(k+1) = g \cdot f$.

In the sequel m, n will be natural numbers. Next we state a number of propositions:

(106) $f^0 = \mathrm{id}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f}$.

(107) $f^{n+1} = (f^n) \cdot f$.

(108) $f^1 = f$.

(109) $f^{n+1} = f \cdot (f^n)$.

(110) $\mathrm{dom}(f^n) \subseteq \mathrm{dom}\, f \cup \mathrm{rng}\, f$ and $\mathrm{rng}(f^n) \subseteq \mathrm{dom}\, f \cup \mathrm{rng}\, f$.

(111) If $n \neq 0$, then $\mathrm{dom}(f^n) \subseteq \mathrm{dom}\, f$ and $\mathrm{rng}(f^n) \subseteq \mathrm{rng}\, f$.

(112) If $\mathrm{rng}\, f \subseteq \mathrm{dom}\, f$, then $\mathrm{dom}(f^n) = \mathrm{dom}\, f$ and $\mathrm{rng}(f^n) \subseteq \mathrm{dom}\, f$.

(113) $(f^n) \cdot \mathrm{id}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f} = f^n$.

(114) $\mathrm{id}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f} \cdot (f^n) = f^n$.

(115) $(f^n) \cdot (f^m) = f^{n+m}$.

(116) If $n \neq 0$, then $(f^m)^n = f^{m \cdot n}$.

(117) If $\mathrm{rng}\, f \subseteq \mathrm{dom}\, f$, then $(f^m)^n = f^{m \cdot n}$.

(118) $\square^n = \square$.

(119) ${\mathrm{id}_X}^n = \mathrm{id}_X$.

(120) If $\mathrm{rng}\, f \cap \mathrm{dom}\, f = \emptyset$, then $f^2 = \square$.

(121) For every function f from X into X holds f^n is a function from X into X.

(122) For every function f from X into X holds $f^0 = \mathrm{id}_X$.

(123) For every function f from X into X holds $(f^m)^n = f^{m \cdot n}$.

(124) For every partial function f from X to X holds f^n is a partial function from X to X.

(125) If $n \neq 0$ and $a \in X$ and $f = X \longmapsto a$, then $f^n = f$.

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