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Curried and Uncurried Functions

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Summary. In the article following functors are introduced: the projections of subsets of the Cartesian product, the functor which for every function $f : X \times Y \to Z$ gives some curried function $(X \to (Y \to Z))$, and the functor which from curried functions makes uncurried functions. Some of their properties and some properties of the set of all functions from a set into a set are also shown.

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The papers [8], [3], [2], [4], [9], [1], [6], [7], and [5] provide the terminology and notation for this paper. We follow a convention: X, Y, Z, X_1, X_2, Y_1, Y_2 are sets, f, g, f_1, f_2 are functions, and x, y, z, t are arbitrary. The scheme *LambdaFS* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ and states that:

there exists f such that $\operatorname{dom} f = \mathcal{A}$ and for every g such that $g \in \mathcal{A}$ holds $f(g) = \mathcal{F}(g)$

for all values of the parameters.

We now state a proposition

(1) $\curvearrowleft\Box = \Box$.

We now define two new functors. Let us consider X. The functor $\pi_1(X)$ yields a set and is defined as follows:

$x \in \pi_1(X)$ if and only if there exists y such that $\langle x, y \rangle \in X$.

The functor $\pi_2(X)$ yields a set and is defined as follows:

$y \in \pi_2(X)$ if and only if there exists x such that $\langle x, y \rangle \in X$.

The following propositions are true:

(2) $Z = \pi_1(X)$ if and only if for every x holds $x \in Z$ if and only if there exists y such that $\langle x, y \rangle \in X$.

(3) $Z = \pi_2(X)$ if and only if for every y holds $y \in Z$ if and only if there exists x such that $\langle x, y \rangle \in X$.

(4) If $\langle x, y \rangle \in X$, then $x \in \pi_1(X)$ and $y \in \pi_2(X)$.

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(5) If $X \subseteq Y$, then $\pi_1(X) \subseteq \pi_1(Y)$ and $\pi_2(X) \subseteq \pi_2(Y)$.

(6) $\pi_1(X \cup Y) = \pi_1(X) \cup \pi_1(Y)$ and $\pi_2(X \cup Y) = \pi_2(X) \cup \pi_2(Y)$.

(7) $\pi_1(X \cap Y) \subseteq \pi_1(X) \cap \pi_1(Y)$ and $\pi_2(X \cap Y) \subseteq \pi_2(X) \cap \pi_2(Y)$.

(8) $\pi_1(X) \setminus \pi_1(Y) \subseteq \pi_1(X \setminus Y)$ and $\pi_2(X) \setminus \pi_2(Y) \subseteq \pi_2(X \setminus Y)$.

(9) $\pi_1(X) \dot{-} \pi_1(Y) \subseteq \pi_1(X \dot{-} Y)$ and $\pi_2(X) \dot{-} \pi_2(Y) \subseteq \pi_2(X \dot{-} Y)$.

(10) $\pi_1(\emptyset) = \emptyset$ and $\pi_2(\emptyset) = \emptyset$.

(11) If $Y \neq \emptyset$ or $[\!: X, Y :\!] \neq \emptyset$ or $[\!: Y, X :\!] \neq \emptyset$, then $\pi_1([\!: X, Y :\!]) = X$ and $\pi_2([\!: Y, X :\!]) = X$.

(12) $\pi_1([\!: X, Y :\!]) \subseteq X$ and $\pi_2([\!: X, Y :\!]) \subseteq Y$.

(13) If $Z \subseteq [\!: X, Y :\!]$, then $\pi_1(Z) \subseteq X$ and $\pi_2(Z) \subseteq Y$.

(14) $\pi_1([\!: X, \{x\} :\!]) = X$ and $\pi_2([\!: \{x\}, X :\!]) = X$ and $\pi_1([\!: X, \{x, y\} :\!]) = X$ and $\pi_2([\!: \{x, y\}, X :\!]) = X$.

(15) $\pi_1(\{\langle x, y \rangle\}) = \{x\}$ and $\pi_2(\{\langle x, y \rangle\}) = \{y\}$.

(16) $\pi_1(\{\langle x, y \rangle, \langle z, t \rangle\}) = \{x, z\}$ and $\pi_2(\{\langle x, y \rangle, \langle z, t \rangle\}) = \{y, t\}$.

(17) If for no x, y holds $\langle x, y \rangle \in X$, then $\pi_1(X) = \emptyset$ and $\pi_2(X) = \emptyset$.

(18) If $\pi_1(X) = \emptyset$ or $\pi_2(X) = \emptyset$, then for no x, y holds $\langle x, y \rangle \in X$.

(19) $\pi_1(X) = \emptyset$ if and only if $\pi_2(X) = \emptyset$.

(20) $\pi_1(\mathrm{dom}\, f) = \pi_2(\mathrm{dom}(\curvearrowleft f))$ and $\pi_2(\mathrm{dom}\, f) = \pi_1(\mathrm{dom}(\curvearrowleft f))$.

(21) $\pi_1(\mathrm{graph}\, f) = \mathrm{dom}\, f$ and $\pi_2(\mathrm{graph}\, f) = \mathrm{rng}\, f$.

We now define two new functors. Let us consider f. The functor $\mathrm{curry}\, f$ yielding a function, is defined by:

(i) $\mathrm{dom}(\mathrm{curry}\, f) = \pi_1(\mathrm{dom}\, f)$,

(ii) for every x such that $x \in \pi_1(\mathrm{dom}\, f)$ there exists g such that $(\mathrm{curry}\, f)(x) = g$ and $\mathrm{dom}\, g = \pi_2(\mathrm{dom}\, f \cap [\!: \{x\}, \pi_2(\mathrm{dom}\, f) :\!])$ and for every y such that $y \in \mathrm{dom}\, g$ holds $g(y) = f(\langle x, y \rangle)$.

The functor $\mathrm{uncurry}\, f$ yields a function and is defined as follows:

(i) for every t holds $t \in \mathrm{dom}(\mathrm{uncurry}\, f)$ if and only if there exist x, g, y such that $t = \langle x, y \rangle$ and $x \in \mathrm{dom}\, f$ and $g = f(x)$ and $y \in \mathrm{dom}\, g$,

(ii) for all x, g such that $x \in \mathrm{dom}(\mathrm{uncurry}\, f)$ and $g = f(x_{\mathbf{1}})$ holds $(\mathrm{uncurry}\, f)(x) = g(x_{\mathbf{2}})$.

We now define two new functors. Let us consider f. The functor $\mathrm{curry}'\, f$ yields a function and is defined as follows:

$\mathrm{curry}'\, f = \mathrm{curry}(\curvearrowleft f)$.

The functor $\mathrm{uncurry}'\, f$ yielding a function, is defined by:

$\mathrm{uncurry}'\, f = \curvearrowleft(\mathrm{uncurry}\, f)$.

The following propositions are true:

(22) Let F be a function. Then $F = \mathrm{curry}\, f$ if and only if the following conditions are satisfied:

(i) $\mathrm{dom}\, F = \pi_1(\mathrm{dom}\, f)$,

(ii) for every x such that $x \in \pi_1(\mathrm{dom}\, f)$ there exists g such that $F(x) = g$ and $\mathrm{dom}\, g = \pi_2(\mathrm{dom}\, f \cap [\!: \{x\}, \pi_2(\mathrm{dom}\, f) :\!])$ and for every y such that $y \in \mathrm{dom}\, g$ holds $g(y) = f(\langle x, y \rangle)$.

(23) $\mathrm{curry}'\, f = \mathrm{curry}(\curvearrowleft f)$.

(24) Let F be a function. Then $F = \mathrm{uncurry}\, f$ if and only if the following conditions are satisfied:

(i) for every t holds $t \in \mathrm{dom}\, F$ if and only if there exist x, g, y such that $t = \langle x, y \rangle$ and $x \in \mathrm{dom}\, f$ and $g = f(x)$ and $y \in \mathrm{dom}\, g$,

(ii) for all x, g such that $x \in \mathrm{dom}\, F$ and $g = f(x_{\mathbf{1}})$ holds $F(x) = g(x_{\mathbf{2}})$.

(25) $\mathrm{uncurry}'\, f = \curvearrowleft(\mathrm{uncurry}\, f)$.

(26) If $\langle x, y \rangle \in \mathrm{dom}\, f$, then $x \in \mathrm{dom}(\mathrm{curry}\, f)$ and $\mathrm{curry}\, f(x)$ is a function.

(27) If $\langle x, y \rangle \in \mathrm{dom}\, f$ and $g = \mathrm{curry}\, f(x)$, then $y \in \mathrm{dom}\, g$ and $g(y) = f(\langle x, y \rangle)$.

(28) If $\langle x, y \rangle \in \mathrm{dom}\, f$, then $y \in \mathrm{dom}(\mathrm{curry}'\, f)$ and $\mathrm{curry}'\, f(y)$ is a function.

(29) If $\langle x, y \rangle \in \mathrm{dom}\, f$ and $g = \mathrm{curry}'\, f(y)$, then $x \in \mathrm{dom}\, g$ and $g(x) = f(\langle x, y \rangle)$.

(30) $\mathrm{dom}(\mathrm{curry}'\, f) = \pi_2(\mathrm{dom}\, f)$.

(31) If $[\!: X, Y :\!] \neq \emptyset$ and $\mathrm{dom}\, f = [\!: X, Y :\!]$, then $\mathrm{dom}(\mathrm{curry}\, f) = X$ and $\mathrm{dom}(\mathrm{curry}'\, f) = Y$.

(32) If $\mathrm{dom}\, f \subseteq [\!: X, Y :\!]$, then $\mathrm{dom}(\mathrm{curry}\, f) \subseteq X$ and $\mathrm{dom}(\mathrm{curry}'\, f) \subseteq Y$.

(33) If $\mathrm{rng}\, f \subseteq Y^X$, then $\mathrm{dom}(\mathrm{uncurry}\, f) = [\!: \mathrm{dom}\, f, X :\!]$ and $\mathrm{dom}(\mathrm{uncurry}'\, f) = [\!: X, \mathrm{dom}\, f :\!]$.

(34) If for no x, y holds $\langle x, y \rangle \in \mathrm{dom}\, f$, then $\mathrm{curry}\, f = \square$ and $\mathrm{curry}'\, f = \square$.

(35) If for no x holds $x \in \mathrm{dom}\, f$ and $f(x)$ is a function, then $\mathrm{uncurry}\, f = \square$ and $\mathrm{uncurry}'\, f = \square$.

(36) Suppose $[\!: X, Y :\!] \neq \emptyset$ and $\mathrm{dom}\, f = [\!: X, Y :\!]$ and $x \in X$. Then there exists g such that $\mathrm{curry}\, f(x) = g$ and $\mathrm{dom}\, g = Y$ and $\mathrm{rng}\, g \subseteq \mathrm{rng}\, f$ and for every y such that $y \in Y$ holds $g(y) = f(\langle x, y \rangle)$.

(37) If $x \in \mathrm{dom}(\mathrm{curry}\, f)$, then $\mathrm{curry}\, f(x)$ is a function.

(38) Suppose $x \in \mathrm{dom}(\mathrm{curry}\, f)$ and $g = \mathrm{curry}\, f(x)$. Then

(i) $\mathrm{dom}\, g = \pi_2(\mathrm{dom}\, f \cap [\!: \{x\}, \pi_2(\mathrm{dom}\, f) :\!])$,

(ii) $\mathrm{dom}\, g \subseteq \pi_2(\mathrm{dom}\, f)$,

(iii) $\mathrm{rng}\, g \subseteq \mathrm{rng}\, f$,

(iv) for every y such that $y \in \mathrm{dom}\, g$ holds $g(y) = f(\langle x, y \rangle)$ and $\langle x, y \rangle \in \mathrm{dom}\, f$.

(39) Suppose $[\!: X, Y :\!] \neq \emptyset$ and $\mathrm{dom}\, f = [\!: X, Y :\!]$ and $y \in Y$. Then there exists g such that $\mathrm{curry}'\, f(y) = g$ and $\mathrm{dom}\, g = X$ and $\mathrm{rng}\, g \subseteq \mathrm{rng}\, f$ and for every x such that $x \in X$ holds $g(x) = f(\langle x, y \rangle)$.

(40) If $x \in \mathrm{dom}(\mathrm{curry}'\, f)$, then $\mathrm{curry}'\, f(x)$ is a function.

(41) Suppose $x \in \mathrm{dom}(\mathrm{curry}'\, f)$ and $g = \mathrm{curry}'\, f(x)$. Then

(i) $\mathrm{dom}\, g = \pi_1(\mathrm{dom}\, f \cap [\!: \pi_1(\mathrm{dom}\, f), \{x\} :\!])$,

(ii) $\mathrm{dom}\, g \subseteq \pi_1(\mathrm{dom}\, f)$,

(iii) $\mathrm{rng}\, g \subseteq \mathrm{rng}\, f$,

(iv) for every y such that $y \in \mathrm{dom}\, g$ holds $g(y) = f(\langle y, x \rangle)$ and $\langle y, x \rangle \in \mathrm{dom}\, f$.

(42) If $\operatorname{dom} f = [: X, Y :]$, then $\operatorname{rng}(\operatorname{curry} f) \subseteq (\operatorname{rng} f)^Y$ and $\operatorname{rng}(\operatorname{curry}' f) \subseteq (\operatorname{rng} f)^X$.

(43) $\operatorname{rng}(\operatorname{curry} f) \subseteq \pi_2(\operatorname{dom} f) \dot{\rightarrow} (\operatorname{rng} f)$ and $\operatorname{rng}(\operatorname{curry}' f) \subseteq \pi_1(\operatorname{dom} f) \dot{\rightarrow} (\operatorname{rng} f)$.

(44) If $\operatorname{rng} f \subseteq X \dot{\rightarrow} Y$, then $\operatorname{dom}(\operatorname{uncurry} f) \subseteq [: \operatorname{dom} f, X :]$ and $\operatorname{dom}(\operatorname{uncurry}' f) \subseteq [: X, \operatorname{dom} f :]$.

(45) If $x \in \operatorname{dom} f$ and $g = f(x)$ and $y \in \operatorname{dom} g$, then $\langle x, y \rangle \in \operatorname{dom}(\operatorname{uncurry} f)$ and $\operatorname{uncurry} f(\langle x, y \rangle) = g(y)$ and $g(y) \in \operatorname{rng}(\operatorname{uncurry} f)$.

(46) If $x \in \operatorname{dom} f$ and $g = f(x)$ and $y \in \operatorname{dom} g$, then $\langle y, x \rangle \in \operatorname{dom}(\operatorname{uncurry}' f)$ and $\operatorname{uncurry}' f(\langle y, x \rangle) = g(y)$ and $g(y) \in \operatorname{rng}(\operatorname{uncurry}' f)$.

(47) If $\operatorname{rng} f \subseteq X \dot{\rightarrow} Y$, then $\operatorname{rng}(\operatorname{uncurry} f) \subseteq Y$ and $\operatorname{rng}(\operatorname{uncurry}' f) \subseteq Y$.

(48) If $\operatorname{rng} f \subseteq Y^X$, then $\operatorname{rng}(\operatorname{uncurry} f) \subseteq Y$ and $\operatorname{rng}(\operatorname{uncurry}' f) \subseteq Y$.

(49) $\operatorname{curry} \square = \square$ and $\operatorname{curry}' \square = \square$.

(50) $\operatorname{uncurry} \square = \square$ and $\operatorname{uncurry}' \square = \square$.

(51) If $\operatorname{dom} f_1 = [: X, Y :]$ and $\operatorname{dom} f_2 = [: X, Y :]$ and $\operatorname{curry} f_1 = \operatorname{curry} f_2$, then $f_1 = f_2$.

(52) If $\operatorname{dom} f_1 = [: X, Y :]$ and $\operatorname{dom} f_2 = [: X, Y :]$ and $\operatorname{curry}' f_1 = \operatorname{curry}' f_2$, then $f_1 = f_2$.

(53) If $\operatorname{rng} f_1 \subseteq Y^X$ and $\operatorname{rng} f_2 \subseteq Y^X$ and $X \neq \emptyset$ and $\operatorname{uncurry} f_1 = \operatorname{uncurry} f_2$, then $f_1 = f_2$.

(54) If $\operatorname{rng} f_1 \subseteq Y^X$ and $\operatorname{rng} f_2 \subseteq Y^X$ and $X \neq \emptyset$ and $\operatorname{uncurry}' f_1 = \operatorname{uncurry}' f_2$, then $f_1 = f_2$.

(55) If $\operatorname{rng} f \subseteq Y^X$ and $X \neq \emptyset$, then $\operatorname{curry}(\operatorname{uncurry} f) = f$ and $\operatorname{curry}'(\operatorname{uncurry}' f) = f$.

(56) If $\operatorname{dom} f = [: X, Y :]$, then $\operatorname{uncurry}(\operatorname{curry} f) = f$ and $\operatorname{uncurry}'(\operatorname{curry}' f) = f$.

(57) If $\operatorname{dom} f \subseteq [: X, Y :]$, then $\operatorname{uncurry}(\operatorname{curry} f) = f$ and $\operatorname{uncurry}'(\operatorname{curry}' f) = f$.

(58) If $\operatorname{rng} f \subseteq X \dot{\rightarrow} Y$ and $\square \notin \operatorname{rng} f$, then $\operatorname{curry}(\operatorname{uncurry} f) = f$ and $\operatorname{curry}'(\operatorname{uncurry}' f) = f$.

(59) If $\operatorname{dom} f_1 \subseteq [: X, Y :]$ and $\operatorname{dom} f_2 \subseteq [: X, Y :]$ and $\operatorname{curry} f_1 = \operatorname{curry} f_2$, then $f_1 = f_2$.

(60) If $\operatorname{dom} f_1 \subseteq [: X, Y :]$ and $\operatorname{dom} f_2 \subseteq [: X, Y :]$ and $\operatorname{curry}' f_1 = \operatorname{curry}' f_2$, then $f_1 = f_2$.

(61) If $\operatorname{rng} f_1 \subseteq X \dot{\rightarrow} Y$ and $\operatorname{rng} f_2 \subseteq X \dot{\rightarrow} Y$ and $\square \notin \operatorname{rng} f_1$ and $\square \notin \operatorname{rng} f_2$ and $\operatorname{uncurry} f_1 = \operatorname{uncurry} f_2$, then $f_1 = f_2$.

(62) If $\operatorname{rng} f_1 \subseteq X \dot{\rightarrow} Y$ and $\operatorname{rng} f_2 \subseteq X \dot{\rightarrow} Y$ and $\square \notin \operatorname{rng} f_1$ and $\square \notin \operatorname{rng} f_2$ and $\operatorname{uncurry}' f_1 = \operatorname{uncurry}' f_2$, then $f_1 = f_2$.

(63) If $X \subseteq Y$, then $X^Z \subseteq Y^Z$.

(64) $X^{\emptyset} = \{\square\}$.

(65) $X \approx X^{\{x\}}$ and $\overline{\overline{X}} = \overline{\overline{X^{\{x\}}}}$.

(66) $\{x\}^X = \{X \longmapsto x\}$.

(67) If $X_1 \approx Y_1$ and $X_2 \approx Y_2$, then ${X_2}^{X_1} \approx {Y_2}^{Y_1}$ and $\overline{\overline{{X_2}^{X_1}}} = \overline{\overline{{Y_2}^{Y_1}}}$.

(68) If $\overline{\overline{X_1}} = \overline{\overline{Y_1}}$ and $\overline{\overline{X_2}} = \overline{\overline{Y_2}}$, then $\overline{\overline{{X_2}^{X_1}}} = \overline{\overline{{Y_2}^{Y_1}}}$.

(69) If $X_1 \cap X_2 = \emptyset$, then $X^{X_1 \cup X_2} \approx [\!: X^{X_1}, X^{X_2} :\!]$ and $\overline{\overline{X^{X_1 \cup X_2}}} = \overline{\overline{[\!: X^{X_1}, X^{X_2} :\!]}}$.

(70) $Z^{[\!: X, Y :\!]} \approx (Z^Y)^X$ and $\overline{\overline{Z^{[\!: X, Y :\!]}}} = \overline{\overline{(Z^Y)^X}}$.

(71) $[\!: X, Y :\!]^Z \approx [\!: X^Z, Y^Z :\!]$ and $\overline{\overline{[\!: X, Y :\!]^Z}} = \overline{\overline{[\!: X^Z, Y^Z :\!]}}$.

(72) If $x \neq y$, then $\{x, y\}^X \approx 2^X$ and $\overline{\overline{\{x, y\}^X}} = \overline{\overline{2^X}}$.

(73) If $x \neq y$, then $X^{\{x,y\}} \approx [\!: X, X :\!]$ and $\overline{\overline{X^{\{x,y\}}}} = \overline{\overline{[\!: X, X :\!]}}$.

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