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# Integers

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**Summary.** In the article the following concepts were introduced: the set of integers ($\mathbb{Z}$) and its elements (integers), congruences ($i_1 \equiv i_2 (\bmod\, i_3)$), the ceiling and floor functors ($\lceil x \rceil$ and $\lfloor x \rfloor$), also the fraction part of a real number (frac), the integer division ($\div$) and remainder of integer division (mod). The following schemes were also included: the separation scheme (*SepInt*), the schemes of integer induction (*Int_Ind_Down*, *Int_Ind_Up*, *Int_Ind_Full*), the minimum (*Int_Min*) and maximum (*Int_Max*) schemes (the existence of minimum and maximum integers enjoying a given property).

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The papers [2], and [1] provide the notation and terminology for this paper. For simplicity we follow a convention: $x$ is arbitrary, $k$, $n_1$, $n_2$ denote natural numbers, $r$, $r_1$, $r_2$ denote real numbers, and $D$ denotes a non-empty set. The following propositions are true:

(1) $(r + r_1) - r_2 = (r - r_2) + r_1$.

(2) $(-r_1) + r_2 = r_2 - r_1$.

(3) $r_1 = ((-r_2) + r_1) + r_2$ and $r_1 = r_2 + ((-r_2) + r_1)$ and $r_1 = r_2 + (r_1 - r_2)$ and $r_1 = (r_2 + r_1) - r_2$.

(4) $(r_1 - r_2) + r_2 = r_1$ and $(r_1 + r_2) - r_2 = r_1$.

(5) $r_1 \leq r_2$ if and only if $r_1 < r_2$ or $r_1 = r_2$.

The non-empty set $\mathbb{Z}$ is defined by:
$x \in \mathbb{Z}$ if and only if there exists $k$ such that $x = k$ or $x = -k$.

One can prove the following proposition

(6) For every $x$ holds $x \in D$ if and only if there exists $k$ such that $x = k$ or $x = -k$ if and only if $D = \mathbb{Z}$.

A real number is called an integer if:
it is an element of $\mathbb{Z}$.

The following propositions are true:




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(7) $r$ is an integer if and only if $r$ is an element of $\mathbb{Z}$.

(8) $r$ is an integer if and only if there exists $k$ such that $r = k$ or $r = -k$.

(9) If $x$ is a natural number, then $x$ is an integer.

(10) 0 is an integer and 1 is an integer.

(11) If $x \in \mathbb{Z}$, then $x \in \mathbb{R}$.

(12) $x$ is an integer if and only if $x \in \mathbb{Z}$.

(13) $x$ is an integer if and only if $x$ is an element of $\mathbb{Z}$.

(14) $\mathbb{N} \subseteq \mathbb{Z}$.

(15) $\mathbb{Z} \subseteq \mathbb{R}$.

In the sequel $i_0$, $i_1$, $i_2$, $i_3$, $i_4$, $i_5$ are integers. Let $i_1$, $i_2$ be integers. Then $i_1 + i_2$ is an integer. Then $i_1 \cdot i_2$ is an integer.

Let $i_0$ be an integer. Then $-i_0$ is an integer.

Let $i_1$, $i_2$ be integers. Then $i_1 - i_2$ is an integer.

Let $n$ be a natural number. Then $-n$ is an integer. Let $i_1$ be an integer. Then $n + i_1$ is an integer. Then $n \cdot i_1$ is an integer. Then $n - i_1$ is an integer.

Let $i_1$ be an integer, and let $n$ be a natural number. Then $i_1 + n$ is an integer. Then $i_1 \cdot n$ is an integer. Then $i_1 - n$ is an integer.

Let us consider $n_1$, $n_2$. Then $n_1 - n_2$ is an integer.

We now state a number of propositions:

(16) If $0 \leq i_0$, then $i_0$ is a natural number.

(17) If $r$ is an integer, then $r + 1$ is an integer and $r - 1$ is an integer.

(18) If $i_2 \leq i_1$, then $i_1 - i_2$ is a natural number.

(19) If $i_1 + k = i_2$ or $k + i_1 = i_2$, then $i_1 \leq i_2$.

(20) If $i_0 < i_1$, then $i_0 + 1 \leq i_1$ and $1 + i_0 \leq i_1$.

(21) If $i_1 < 0$, then $i_1 \leq -1$.

(22) $i_1 \cdot i_2 = 1$ if and only if $i_1 = 1$ and $i_2 = 1$ or $i_1 = -1$ and $i_2 = -1$.

(23) $i_1 \cdot i_2 = -1$ if and only if $i_1 = -1$ and $i_2 = 1$ or $i_1 = 1$ and $i_2 = -1$.

(24) If $i_0 \neq 0$, then $i_1 \neq i_1 + i_0$.

(25) $i_1 < i_1 + 1$.

(26) $i_1 - 1 < i_1$.

(27) For no $i_0$ holds for every $i_1$ holds $i_0 < i_1$.

(28) For no $i_0$ holds for every $i_1$ holds $i_1 < i_0$.

In the article we present several logical schemes. The scheme *SepInt* deals with a unary predicate $\mathcal{P}$, and states that:

there exists a subset $X$ of $\mathbb{Z}$ such that for every integer $x$ holds $x \in X$ if and only if $\mathcal{P}[x]$

for all values of the parameter.

The scheme *Int_Ind_Up* concerns an integer $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every $i_0$ such that $\mathcal{A} \leq i_0$ holds $\mathcal{P}[i_0]$

provided the following conditions are fulfilled:

- $\mathcal{P}[\mathcal{A}]$,
- for every $i_2$ such that $\mathcal{A} \leq i_2$ holds if $\mathcal{P}[i_2]$, then $\mathcal{P}[i_2 + 1]$.

The scheme *Int_Ind_Down* deals with an integer $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every $i_0$ such that $i_0 \leq \mathcal{A}$ holds $\mathcal{P}[i_0]$

provided the parameters fulfill the following conditions:

- $\mathcal{P}[\mathcal{A}]$,
- for every $i_2$ such that $i_2 \leq \mathcal{A}$ holds if $\mathcal{P}[i_2]$, then $\mathcal{P}[i_2 - 1]$.

The scheme *Int_Ind_Full* deals with an integer $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every $i_0$ holds $\mathcal{P}[i_0]$

provided the following requirements are fulfilled:

- $\mathcal{P}[\mathcal{A}]$,
- for every $i_2$ such that $\mathcal{P}[i_2]$ holds $\mathcal{P}[i_2 - 1]$ and $\mathcal{P}[i_2 + 1]$.

The scheme *Int_Min* concerns an integer $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists $i_0$ such that $\mathcal{P}[i_0]$ and for every $i_1$ such that $\mathcal{P}[i_1]$ holds $i_0 \leq i_1$

provided the following conditions are satisfied:

- for every $i_1$ such that $\mathcal{P}[i_1]$ holds $\mathcal{A} \leq i_1$,
- there exists $i_1$ such that $\mathcal{P}[i_1]$.

The scheme *Int_Max* deals with an integer $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists $i_0$ such that $\mathcal{P}[i_0]$ and for every $i_1$ such that $\mathcal{P}[i_1]$ holds $i_1 \leq i_0$

provided the parameters satisfy the following conditions:

- for every $i_1$ such that $\mathcal{P}[i_1]$ holds $i_1 \leq \mathcal{A}$,
- there exists $i_1$ such that $\mathcal{P}[i_1]$.

Let us consider $r$. Then $\operatorname{sgn} r$ is an integer.

We now state two propositions:

(29) $\operatorname{sgn} r = 1$ or $\operatorname{sgn} r = -1$ or $\operatorname{sgn} r = 0$.

(30) $|r| = r$ or $|r| = -r$.

Let us consider $i_0$. Then $|i_0|$ is an integer.

Let $i_1$, $i_2$, $i_3$ be integers. The predicate $i_1 \equiv i_2 (\operatorname{mod} i_3)$ is defined by:

there exists $i_4$ such that $i_3 \cdot i_4 = i_1 - i_2$.

We now state a number of propositions:

(31) $i_1 \equiv i_2 (\operatorname{mod} i_3)$ if and only if there exists an integer $i_4$ such that $i_3 \cdot i_4 = i_1 - i_2$.

(32) $i_1 \equiv i_1 (\operatorname{mod} i_2)$.

(33) If $i_2 = 0$, then $i_1 \equiv i_2 (\operatorname{mod} i_1)$ and $i_2 \equiv i_1 (\operatorname{mod} i_1)$.

(34) If $i_3 = 1$, then $i_1 \equiv i_2 (\operatorname{mod} i_3)$.

(35) If $i_1 \equiv i_2 (\operatorname{mod} i_3)$, then $i_2 \equiv i_1 (\operatorname{mod} i_3)$.

(36) If $i_1 \equiv i_2 (\operatorname{mod} i_5)$ and $i_2 \equiv i_3 (\operatorname{mod} i_5)$, then $i_1 \equiv i_3 (\operatorname{mod} i_5)$.

(37) If $i_1 \equiv i_2 (\operatorname{mod} i_5)$ and $i_3 \equiv i_4 (\operatorname{mod} i_5)$, then $i_1 + i_3 \equiv i_2 + i_4 (\operatorname{mod} i_5)$.

(38) If $i_1 \equiv i_2 (\mathrm{mod}\, i_5)$ and $i_3 \equiv i_4 (\mathrm{mod}\, i_5)$, then $i_1 - i_3 \equiv i_2 - i_4 (\mathrm{mod}\, i_5)$.

(39) If $i_1 \equiv i_2 (\mathrm{mod}\, i_5)$ and $i_3 \equiv i_4 (\mathrm{mod}\, i_5)$, then $i_1 \cdot i_3 \equiv i_2 \cdot i_4 (\mathrm{mod}\, i_5)$.

(40) $i_1 + i_2 \equiv i_3 (\mathrm{mod}\, i_5)$ if and only if $i_1 \equiv i_3 - i_2 (\mathrm{mod}\, i_5)$.

(41) If $i_4 \cdot i_5 = i_3$, then if $i_1 \equiv i_2 (\mathrm{mod}\, i_3)$, then $i_1 \equiv i_2 (\mathrm{mod}\, i_4)$.

(42) $i_1 \equiv i_2 (\mathrm{mod}\, i_5)$ if and only if $i_1 + i_5 \equiv i_2 (\mathrm{mod}\, i_5)$.

(43) $i_1 \equiv i_2 (\mathrm{mod}\, i_5)$ if and only if $i_1 - i_5 \equiv i_2 (\mathrm{mod}\, i_5)$.

(44) If $i_1 \leq r$ and $r - 1 < i_1$ and $i_2 \leq r$ and $r - 1 < i_2$, then $i_1 = i_2$.

(45) If $r \leq i_1$ and $i_1 < r + 1$ and $r \leq i_2$ and $i_2 < r + 1$, then $i_1 = i_2$.

Let us consider $r$. The functor $\lfloor r \rfloor$ yielding an integer, is defined as follows: $\lfloor r \rfloor \leq r$ and $r - 1 < \lfloor r \rfloor$.

The following propositions are true:

(46) $i_0 \leq r$ and $r - 1 < i_0$ if and only if $\lfloor r \rfloor = i_0$.

(47) $\lfloor r \rfloor = r$ if and only if $r$ is an integer.

(48) $\lfloor r \rfloor < r$ if and only if $r$ is not an integer.

(49) $\lfloor r \rfloor \leq r$.

(50) $\lfloor r \rfloor - 1 < r$ and $\lfloor r \rfloor < r + 1$.

(51) $\lfloor r \rfloor + i_0 = \lfloor r + i_0 \rfloor$.

(52) $r \leq \lfloor r \rfloor + 1$.

Let us consider $r$. The functor $\lceil r \rceil$ yields an integer and is defined as follows: $r \leq \lceil r \rceil$ and $\lceil r \rceil < r + 1$.

We now state a number of propositions:

(53) $r \leq i_0$ and $i_0 < r + 1$ if and only if $\lceil r \rceil = i_0$.

(54) $\lceil r \rceil = r$ if and only if $r$ is an integer.

(55) $r < \lceil r \rceil$ if and only if $r$ is not an integer.

(56) $r \leq \lceil r \rceil$.

(57) $r - 1 < \lceil r \rceil$ and $r < \lceil r \rceil + 1$.

(58) $\lceil r \rceil + i_0 = \lceil r + i_0 \rceil$.

(59) $\lfloor r \rfloor = \lceil r \rceil$ if and only if $r$ is an integer.

(60) $\lfloor r \rfloor < \lceil r \rceil$ if and only if $r$ is not an integer.

(61) $\lfloor r \rfloor \leq \lceil r \rceil$.

(62) $\lfloor \lceil r \rceil \rfloor = \lceil r \rceil$.

(63) $\lfloor \lfloor r \rfloor \rfloor = \lfloor r \rfloor$.

(64) $\lceil \lceil r \rceil \rceil = \lceil r \rceil$.

(65) $\lceil \lfloor r \rfloor \rceil = \lfloor r \rfloor$.

(66) $\lfloor r \rfloor = \lceil r \rceil$ if and only if $\lfloor r \rfloor + 1 \neq \lceil r \rceil$.

Let us consider $r$. The functor $\mathrm{frac}\, r$ yielding a real number, is defined by: $\mathrm{frac}\, r = r - \lfloor r \rfloor$.

One can prove the following propositions:

(67) $\mathrm{frac}\, r = r - \lfloor r \rfloor$.

(68) $r = \lfloor r \rfloor + \mathrm{frac}\, r$.

(69) $\mathrm{frac}\, r < 1$ and $0 \leq \mathrm{frac}\, r$.

(70) $\lfloor \mathrm{frac}\, r \rfloor = 0$.

(71) $\mathrm{frac}\, r = 0$ if and only if $r$ is an integer.

(72) $0 < \mathrm{frac}\, r$ if and only if $r$ is not an integer.

Let $i_1$, $i_2$ be integers. The functor $i_1 \div i_2$ yields an integer and is defined by:
$i_1 \div i_2 = \lfloor \frac{i_1}{i_2} \rfloor$.

One can prove the following proposition

(73) $i_1 \div i_2 = \lfloor \frac{i_1}{i_2} \rfloor$.

Let $i_1$, $i_2$ be integers. The functor $i_1 \bmod i_2$ yielding an integer, is defined as follows:
$i_1 \bmod i_2 = i_1 - (i_1 \div i_2) \cdot i_2$.

Next we state a proposition

(74) $i_1 \bmod i_2 = i_1 - (i_1 \div i_2) \cdot i_2$.

Let $i_1$, $i_2$ be integers. The predicate $i_1 \mid i_2$ is defined as follows:
there exists $i_3$ such that $i_2 = i_1 \cdot i_3$.

The following proposition is true

(75) $i_1 \mid i_2$ if and only if there exists $i_3$ such that $i_1 \cdot i_3 = i_2$.

## References

[1] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[2] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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