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Variables in Formulae of the First Order Language [1]

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Summary. We develop the first order language defined in [5]. We continue the work done in the article [1]. We prove some schemes of defining by structural induction. We deal with notions of closed subformulae and of still not bound variables in a formula. We introduce the concept of the set of all free variables and the set of all fixed variables occurring in a formula.

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The notation and terminology used in this paper have been introduced in the following articles: [6], [3], [4], [2], [5], and [1]. For simplicity we follow the rules: i, j, k are natural numbers, x is a bound variable, a is a free variable, p, q are elements of WFF, l is a finite sequence of elements of Var, P is a predicate symbol, and V is a non-empty subset of Var. Let F be a function from WFF into WFF, and let us consider p. Then $F(p)$ is an element of WFF.

In the article we present several logical schemes. The scheme *QC_Func_Uniq* deals with a non-empty set $\mathcal{A}$, a function $\mathcal{B}$ from WFF into $\mathcal{A}$, a function $\mathcal{C}$ from WFF into $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{B} = \mathcal{C}$

provided the following conditions are satisfied:

- Given p. Let d_1, d_2 be elements of $\mathcal{A}$. Then
  - (i) if $p =$ VERUM, then $\mathcal{B}(p) = \mathcal{D}$,
  - (ii) if p is atomic, then $\mathcal{B}(p) = \mathcal{F}(p)$,
  - (iii) if p is negative and $d_1 = \mathcal{B}(\mathrm{Arg}(p))$, then $\mathcal{B}(p) = \mathcal{G}(d_1)$,
  - (iv) if p is conjunctive and $d_1 = \mathcal{B}(\mathrm{LeftArg}(p))$ and

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$d_2 = \mathcal{B}(\mathrm{RightArg}(p))$,
then $\mathcal{B}(p) = \mathcal{H}(d_1, d_2)$,

(v) if p is universal and $d_1 = \mathcal{B}(\mathrm{Scope}(p))$, then $\mathcal{B}(p) = \mathcal{I}(p, d_1)$,

- Given p. Let d_1, d_2 be elements of $\mathcal{A}$. Then
  (i) if $p = \mathrm{VERUM}$, then $\mathcal{C}(p) = \mathcal{D}$,
  (ii) if p is atomic, then $\mathcal{C}(p) = \mathcal{F}(p)$,
  (iii) if p is negative and $d_1 = \mathcal{C}(\mathrm{Arg}(p))$, then $\mathcal{C}(p) = \mathcal{G}(d_1)$,
  (iv) if p is conjunctive and $d_1 = \mathcal{C}(\mathrm{LeftArg}(p))$ and
  $d_2 = \mathcal{C}(\mathrm{RightArg}(p))$,
  then $\mathcal{C}(p) = \mathcal{H}(d_1, d_2)$,
  (v) if p is universal and $d_1 = \mathcal{C}(\mathrm{Scope}(p))$, then $\mathcal{C}(p) = \mathcal{I}(p, d_1)$.

The scheme *QC_Def_D* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, an element $\mathcal{C}$ of WFF, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

(i) there exists an element d of $\mathcal{A}$ and there exists a function F from WFF into $\mathcal{A}$ such that $d = F(\mathcal{C})$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,

(ii) for all elements x_1, x_2 of $\mathcal{A}$ such that there exists a function F from WFF into $\mathcal{A}$ such that $x_1 = F(\mathcal{C})$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$ and there exists a function F from WFF into $\mathcal{A}$ such that $x_2 = F(\mathcal{C})$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$ holds $x_1 = x_2$

for all values of the parameters.

The scheme *QC_D_Result'VERU* deals with a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{F}(\mathrm{VERUM}) = \mathcal{B}$

provided the parameters fulfill the following condition:

- Let p be a formula. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{G}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{H}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{I}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{J}(p, d_1)$.

The scheme *QC_D_Result'atom* concerns a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a formula $\mathcal{C}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{F}(\mathcal{C}) = \mathcal{G}(\mathcal{C})$

provided the following conditions are fulfilled:

- Let p be a formula. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p =$ VERUM, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{G}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{H}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{I}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{J}(p, d_1)$,
- $\mathcal{C}$ is atomic.

The scheme *QC_D_Result'nega* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a formula $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a unary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{J}(\mathcal{C}) = \mathcal{G}(\mathcal{J}(\mathrm{Arg}(\mathcal{C})))$

provided the following requirements are met:

- Let p be a formula. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{J}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p =$ VERUM, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,
- $\mathcal{C}$ is negative.

The scheme *QC_D_Result'conj* concerns a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$, and a formula $\mathcal{C}$ and states that:

for all elements d_1, d_2 of $\mathcal{A}$ such that $d_1 = \mathcal{J}(\mathrm{LeftArg}(\mathcal{C}))$ and $d_2 = \mathcal{J}(\mathrm{RightArg}(\mathcal{C}))$
holds $\mathcal{J}(\mathcal{C}) = \mathcal{H}(d_1, d_2)$

provided the parameters satisfy the following conditions:

- Let p be a formula. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{J}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p =$ VERUM, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,
- $\mathcal{C}$ is conjunctive.

The scheme *QC_D_Result'univ* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a formula $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a unary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{J}(\mathcal{C}) = \mathcal{I}(\mathcal{C}, \mathcal{J}(\mathrm{Scope}(\mathcal{C})))$

provided the following requirements are fulfilled:

- Let p be a formula. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{J}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p =$ VERUM, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,
- $\mathcal{C}$ is universal.

Let us consider V. The functor $\emptyset_V$ yields an element of 2^V **qua** a non-empty set and is defined as follows:

$\emptyset_V = \emptyset$.

Next we state three propositions:

(1) $\emptyset_V = \emptyset$.

(2) For every k-ary predicate symbol P holds P is a predicate symbol.

(3) P is a $\mathrm{Arity}(P)$-ary predicate symbol.

Let us consider l, V. The functor $\mathrm{variables}_V(l)$ yielding an element of 2^V, is defined by:

$\mathrm{variables}_V(l) = \{l(k) : 1 \leq k \wedge k \leq \mathrm{len}\, l \wedge l(k) \in V\}$.

One can prove the following propositions:

(4) $\mathrm{variables}_V(l) = \{l(k) : 1 \leq k \wedge k \leq \mathrm{len}\, l \wedge l(k) \in V\}$.

(5) $\mathrm{variables}_V(l) \subseteq V$.

(6) $\mathrm{snb}(l) = \mathrm{variables}_{\mathrm{BoundVar}}(l)$.

(7) $\mathrm{snb}(\mathrm{VERUM}) = \emptyset$.

(8) For every formula p such that p is atomic holds $\mathrm{snb}(p) = \mathrm{snb}(\mathrm{Args}(p))$.

(9) For every k-ary predicate symbol P and for every list of variables l of the length k holds $\mathrm{snb}(P[l]) = \mathrm{snb}(l)$.

(10) For every formula p such that p is negative holds $\mathrm{snb}(p) = \mathrm{snb}(\mathrm{Arg}(p))$.

(11) For every formula p holds $\mathrm{snb}(\neg p) = \mathrm{snb}(p)$.

(12) $\mathrm{snb}(\mathrm{FALSUM}) = \emptyset$.

(13) For every formula p such that p is conjunctive holds $\mathrm{snb}(p) = \mathrm{snb}(\mathrm{LeftArg}(p)) \cup \mathrm{snb}(\mathrm{RightArg}(p))$.

(14) For all formulae p, q holds $\mathrm{snb}(p \wedge q) = \mathrm{snb}(p) \cup \mathrm{snb}(q)$.

(15) For every formula p such that p is universal holds $\mathrm{snb}(p) = \mathrm{snb}(\mathrm{Scope}(p)) \setminus \{\mathrm{Bound}(p)\}$.

(16) For every formula p holds $\mathrm{snb}(\forall_x p) = \mathrm{snb}(p) \setminus \{x\}$.

(17) For every formula p such that p is disjunctive holds $\mathrm{snb}(p) = \mathrm{snb}(\mathrm{LeftDisj}(p)) \cup \mathrm{snb}(\mathrm{RightDisj}(p))$.

(18) For all formulae p, q holds $\mathrm{snb}(p \vee q) = \mathrm{snb}(p) \cup \mathrm{snb}(q)$.

(19) For every formula p such that p is conditional holds $\mathrm{snb}(p) = \mathrm{snb}(\mathrm{Antecedent}(p)) \cup \mathrm{snb}(\mathrm{Consequent}(p))$.

(20) For all formulae p, q holds $\mathrm{snb}(p \Rightarrow q) = \mathrm{snb}(p) \cup \mathrm{snb}(q)$.

(21) For every formula p such that p is biconditional holds $\mathrm{snb}(p) = \mathrm{snb}(\mathrm{LeftSide}(p)) \cup \mathrm{snb}(\mathrm{RightSide}(p))$.

(22) For all formulae p, q holds $\mathrm{snb}(p \Leftrightarrow q) = \mathrm{snb}(p) \cup \mathrm{snb}(q)$.

(23) For every formula p holds $\mathrm{snb}(\exists_x p) = \mathrm{snb}(p) \setminus \{x\}$.

(24) VERUM is closed and FALSUM is closed.

(25) For every formula p holds p is closed if and only if $\neg p$ is closed.

(26) For all formulae p, q holds p is closed and q is closed if and only if $p \wedge q$ is closed.

(27) For every formula p holds $\forall_x p$ is closed if and only if $\mathrm{snb}(p) \subseteq \{x\}$.

(28) For every formula p such that p is closed holds $\forall_x p$ is closed.

(29) For all formulae p, q holds p is closed and q is closed if and only if $p \vee q$ is closed.

(30) For all formulae p, q holds p is closed and q is closed if and only if $p \Rightarrow q$ is closed.

(31) For all formulae p, q holds p is closed and q is closed if and only if $p \Leftrightarrow q$ is closed.

(32) For every formula p holds $\exists_x p$ is closed if and only if $\mathrm{snb}(p) \subseteq \{x\}$.

(33) For every formula p such that p is closed holds $\exists_x p$ is closed.

Let us consider V, and let F be a function from WFF into 2^V, and let us consider p. Then $F(p)$ is an element of 2^V.

Let us consider k. The functor x_k yielding a bound variable, is defined as follows:

$x_k = \langle 4, k \rangle$.

One can prove the following propositions:

(34) $x_k = \langle 4, k \rangle$.

(35) If $x_i = x_j$, then $i = j$.

(36) There exists i such that $x_i = x$.

Let us consider k. The functor $\mathbf{a}_k$ yields a free variable and is defined as follows:

$\mathbf{a}_k = \langle 6, k \rangle$.

One can prove the following propositions:

(37) $\mathbf{a}_k = \langle 6, k \rangle$.

(38) If $\mathbf{a}_i = \mathbf{a}_j$, then $i = j$.

(39) There exists i such that $\mathbf{a}_i = a$.

(40) For every element c of FixedVar and for every element a of FreeVar holds $c \neq a$.

(41) For every element c of FixedVar and for every element x of BoundVar holds $c \neq x$.

(42) For every element a of FreeVar and for every element x of BoundVar holds $a \neq x$.

Let us consider V, and let V_1, V_2 be elements of 2^V. Then $V_1 \cup V_2$ is an element of 2^V.

Let D be a non-empty family of sets, and let d be an element of D. The functor $@d$ yields an element of D **qua** a non-empty set and is defined as follows:

$@d = d$.

One can prove the following proposition

(43) For every non-empty family D of sets and for every element d of D holds $@d = d$.

Let D be a non-empty family of sets, and let d be an element of D **qua** a non-empty set. The functor $@d$ yielding an element of D, is defined as follows:

$@d = d$.

We now state a proposition

(44) For every non-empty family D of sets and for every element d of D **qua** a non-empty set holds $@d = d$.

Now we present several schemes. The scheme *QC_Def_SETD* deals with a non-empty family $\mathcal{A}$ of sets, an element $\mathcal{B}$ of $\mathcal{A}$, an element $\mathcal{C}$ of WFF, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

(i) there exists an element d of $\mathcal{A}$ and there exists a function F from WFF into $\mathcal{A}$ such that $d = F(\mathcal{C})$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p =$ VERUM, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,

(ii) for all elements x_1, x_2 of $\mathcal{A}$ such that there exists a function F from WFF into $\mathcal{A}$ such that $x_1 = F(\mathcal{C})$ and for every element p of WFF and for all

elements d_1, d_2 of $\mathcal{A}$ holds if p = VERUM, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$ and there exists a function F from WFF into $\mathcal{A}$ such that $x_2 = F(\mathcal{C})$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if p = VERUM, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$ holds $x_1 = x_2$
for all values of the parameters.

The scheme *QC_SETD_Result'V* concerns a non-empty family $\mathcal{A}$ of sets, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{F}(\mathrm{VERUM}) = \mathcal{B}$

provided the parameters meet the following requirement:

- Let p be an element of WFF. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if p = VERUM, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{G}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{H}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{I}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{J}(p, d_1)$.

The scheme *QC_SETD_Result'a* concerns a non-empty family $\mathcal{A}$ of sets, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, an element $\mathcal{C}$ of WFF, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{F}(\mathcal{C}) = \mathcal{G}(\mathcal{C})$

provided the parameters fulfill the following requirements:

- Let p be an element of WFF. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if p = VERUM, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{G}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{H}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{I}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{J}(p, d_1)$,
- $\mathcal{C}$ is atomic.

The scheme *QC_SETD_Result'n* deals with a non-empty family $\mathcal{A}$ of sets, an element $\mathcal{B}$ of $\mathcal{A}$, an element $\mathcal{C}$ of WFF, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a unary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{J}(\mathcal{C}) = \mathcal{G}(\mathcal{J}(\mathrm{Arg}(\mathcal{C})))$

provided the following requirements are met:

- Let p be an element of WFF. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{J}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p =$ VERUM, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,
- $\mathcal{C}$ is negative.

The scheme *QC_SETD_Result'c* deals with a non-empty family $\mathcal{A}$ of sets, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$, and an element $\mathcal{C}$ of WFF and states that:

for all elements d_1, d_2 of $\mathcal{A}$ such that $d_1 = \mathcal{J}(\mathrm{LeftArg}(\mathcal{C}))$ and $d_2 = \mathcal{J}(\mathrm{RightArg}(\mathcal{C}))$
holds $\mathcal{J}(\mathcal{C}) = \mathcal{H}(d_1, d_2)$

provided the parameters fulfill the following conditions:

- Let p be an element of WFF. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{J}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p =$ VERUM, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,
- $\mathcal{C}$ is conjunctive.

The scheme *QC_SETD_Result'u* deals with a non-empty family $\mathcal{A}$ of sets, an element $\mathcal{B}$ of $\mathcal{A}$, an element $\mathcal{C}$ of WFF, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a unary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{J}(\mathcal{C}) = \mathcal{I}(\mathcal{C}, \mathcal{J}(\mathrm{Scope}(\mathcal{C})))$

provided the parameters meet the following requirements:

- Let p be an element of WFF. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{J}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p =$ VERUM, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and

$d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,

- $\mathcal{C}$ is universal.

Let us consider V, p. The functor $\mathrm{Vars}_V(p)$ yielding an element of 2^V, is defined as follows:

there exists a function F from WFF into 2^V such that $\mathrm{Vars}_V(p) = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of 2^V holds if $p =$ VERUM, then $F(p) = @(\emptyset_V)$ but if p is atomic, then $F(p) = \mathrm{variables}_V(\mathrm{Args}(p))$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = d_1$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = d_1 \cup d_2$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = d_1$.

We now state a number of propositions:

(45) Let X be an element of 2^V. Then $X = \mathrm{Vars}_V(p)$ if and only if there exists a function F from WFF into 2^V such that $X = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of 2^V holds if $p =$ VERUM, then $F(p) = @(\emptyset_V)$ but if p is atomic, then $F(p) = \mathrm{variables}_V(\mathrm{Args}(p))$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = d_1$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = d_1 \cup d_2$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = d_1$.

(46) $\mathrm{Vars}_V(\mathrm{VERUM}) = \emptyset$.

(47) If p is atomic, then $\mathrm{Vars}_V(p) = \mathrm{variables}_V(\mathrm{Args}(p))$ and $\mathrm{Vars}_V(p) = \{\mathrm{Args}(p)(k) : 1 \leq k \wedge k \leq \mathrm{len}\,\mathrm{Args}(p) \wedge \mathrm{Args}(p)(k) \in V\}$.

(48) For every k-ary predicate symbol P and for every list of variables l of the length k holds $\mathrm{Vars}_V(P[l]) = \mathrm{variables}_V(l)$ and $\mathrm{Vars}_V(P[l]) = \{l(i) : 1 \leq i \wedge i \leq \mathrm{len}\, l \wedge l(i) \in V\}$.

(49) If p is negative, then $\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{Arg}(p))$.

(50) $\mathrm{Vars}_V(\neg p) = \mathrm{Vars}_V(p)$.

(51) $\mathrm{Vars}_V(\mathrm{FALSUM}) = \emptyset$.

(52) If p is conjunctive, then
$\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{LeftArg}(p)) \cup \mathrm{Vars}_V(\mathrm{RightArg}(p))$.

(53) $\mathrm{Vars}_V(p \wedge q) = \mathrm{Vars}_V(p) \cup \mathrm{Vars}_V(q)$.

(54) If p is universal, then $\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{Scope}(p))$.

(55) $\mathrm{Vars}_V(\forall_x p) = \mathrm{Vars}_V(p)$.

(56) If p is disjunctive, then
$\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{LeftDisj}(p)) \cup \mathrm{Vars}_V(\mathrm{RightDisj}(p))$.

(57) $\mathrm{Vars}_V(p \vee q) = \mathrm{Vars}_V(p) \cup \mathrm{Vars}_V(q)$.

(58) If p is conditional, then
$\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{Antecedent}(p)) \cup \mathrm{Vars}_V(\mathrm{Consequent}(p))$.

(59) $\mathrm{Vars}_V(p \Rightarrow q) = \mathrm{Vars}_V(p) \cup \mathrm{Vars}_V(q)$.

(60) If p is biconditional, then
$\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{LeftSide}(p)) \cup \mathrm{Vars}_V(\mathrm{RightSide}(p))$.

(61) $\mathrm{Vars}_V(p \Leftrightarrow q) = \mathrm{Vars}_V(p) \cup \mathrm{Vars}_V(q)$.

(62) If p is existential, then $\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{Arg}(\mathrm{Scope}(\mathrm{Arg}(p))))$.

(63) $\mathrm{Vars}_V(\exists_x p) = \mathrm{Vars}_V(p)$.

Let us consider p. The functor $\mathrm{Free}\, p$ yielding an element of 2^{FreeVar}, is defined as follows:

$\mathrm{Free}\, p = \mathrm{Vars}_{\mathrm{FreeVar}}(p)$.

One can prove the following propositions:

(64) $\mathrm{Free}\, p = \mathrm{Vars}_{\mathrm{FreeVar}}(p)$.

(65) $\mathrm{Free}\,\mathrm{VERUM} = \emptyset$.

(66) For every k-ary predicate symbol P and for every list of variables l of the length k holds $\mathrm{Free}(P[l]) = \{l(i) : 1 \leq i \wedge i \leq \mathrm{len}\, l \wedge l(i) \in \mathrm{FreeVar}\}$.

(67) $\mathrm{Free}\, \neg p = \mathrm{Free}\, p$.

(68) $\mathrm{Free}\,\mathrm{FALSUM} = \emptyset$.

(69) $\mathrm{Free}\, p \wedge q = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(70) $\mathrm{Free}\, \forall_x p = \mathrm{Free}\, p$.

(71) $\mathrm{Free}\, p \vee q = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(72) $\mathrm{Free}\, p \Rightarrow q = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(73) $\mathrm{Free}\, p \Leftrightarrow q = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(74) $\mathrm{Free}\, \exists_x p = \mathrm{Free}\, p$.

Let us consider p. The functor $\mathrm{Fixed}\, p$ yielding an element of 2^{FixedVar}, is defined as follows:

$\mathrm{Fixed}\, p = \mathrm{Vars}_{\mathrm{FixedVar}}(p)$.

Next we state a number of propositions:

(75) $\mathrm{Fixed}\, p = \mathrm{Vars}_{\mathrm{FixedVar}}(p)$.

(76) $\mathrm{Fixed}\,\mathrm{VERUM} = \emptyset$.

(77) For every k-ary predicate symbol P and for every list of variables l of the length k holds $\mathrm{Fixed}(P[l]) = \{l(i) : 1 \leq i \wedge i \leq \mathrm{len}\, l \wedge l(i) \in \mathrm{FixedVar}\}$.

(78) $\mathrm{Fixed}\, \neg p = \mathrm{Fixed}\, p$.

(79) $\mathrm{Fixed}\,\mathrm{FALSUM} = \emptyset$.

(80) $\mathrm{Fixed}\, p \wedge q = \mathrm{Fixed}\, p \cup \mathrm{Fixed}\, q$.

(81) $\mathrm{Fixed}(\forall_x p) = \mathrm{Fixed}\, p$.

(82) $\mathrm{Fixed}\, p \vee q = \mathrm{Fixed}\, p \cup \mathrm{Fixed}\, q$.

(83) $\mathrm{Fixed}\, p \Rightarrow q = \mathrm{Fixed}\, p \cup \mathrm{Fixed}\, q$.

(84) $\mathrm{Fixed}\, p \Leftrightarrow q = \mathrm{Fixed}\, p \cup \mathrm{Fixed}\, q$.

(85) $\mathrm{Fixed}(\exists_x p) = \mathrm{Fixed}\, p$.

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