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Linear Combinations in Real Linear Space

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Summary. The article is continuation of [14]. At the beginning we prove some theorems concerning sums of finite sequence of vectors. We introduce the following notions: sum of finite subset of vectors, linear combination, carrier of linear combination, linear combination of elements of a given set of vectors, sum of linear combination. We also show that the set of linear combinations is a real linear space. At the end of article we prove some auxiliary theorems that should be proved in [16], [5], [7], [1] or [8].

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The papers [16], [7], [5], [3], [6], [14], [8], [13], [15], [11], [9], [10], [4], [12], and [2] provide the notation and terminology for this paper. In the article we present several logical schemes. The scheme *LambdaSep1* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that $f(\mathcal{C}) = \mathcal{D}$ and for every element x of $\mathcal{A}$ such that $x \neq \mathcal{C}$ holds $f(x) = \mathcal{F}(x)$
for all values of the parameters.

The scheme *LambdaSep2* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{A}$, an element $\mathcal{E}$ of $\mathcal{B}$, an element $\mathcal{F}$ of $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that $f(\mathcal{C}) = \mathcal{E}$ and $f(\mathcal{D}) = \mathcal{F}$ and for every element x of $\mathcal{A}$ such that $x \neq \mathcal{C}$ and $x \neq \mathcal{D}$ holds $f(x) = \mathcal{F}(x)$
provided the following condition is satisfied:

- $\mathcal{C} \neq \mathcal{D}$.

Let D be a non-empty set. Then $\emptyset_D$ is a subset of D.

For simplicity we follow the rules: X, Y are sets, x is arbitrary, i, k, n are natural numbers, S is an RLS structure, V is a real linear space, u, v, v_1, v_2, v_3 are vectors of V, a, b, r are real numbers, F, G, H are finite sequences of elements of the vectors of V, A, B are subsets of the vectors of V, and f is a

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function from the vectors of V into $\mathbb{R}$. Let us consider S, and let v be an element of the vectors of S. The functor $@v$ yielding a vector of S, is defined as follows:

$@v = v$.

One can prove the following proposition

(1) For every element v of the vectors of V holds $v = @v$.

Let us consider S, x. Let us assume that $x \in S$. The functor x^S yielding a vector of S, is defined as follows:

$x^S = x$.

The following propositions are true:

(2) If $x \in S$, then $x^S = x$.

(3) For every vector v of S holds $v^S = v$.

(4) If $\operatorname{len} F = \operatorname{len} G$ and $\operatorname{len} F = \operatorname{len} H$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $H(k) = @(\pi_k F) + @(\pi_k G)$, then $\sum H = \sum F + \sum G$.

(5) If $\operatorname{len} F = \operatorname{len} G$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $G(k) = a \cdot @(\pi_k F)$, then $\sum G = a \cdot \sum F$.

(6) If $\operatorname{len} F = \operatorname{len} G$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $G(k) = -@(\pi_k F)$, then $\sum G = -\sum F$.

(7) If $\operatorname{len} F = \operatorname{len} G$ and $\operatorname{len} F = \operatorname{len} H$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $H(k) = @(\pi_k F) - @(\pi_k G)$, then $\sum H = \sum F - \sum G$.

(8) For all F, G and for every permutation f of $\operatorname{dom} F$ such that $\operatorname{len} F = \operatorname{len} G$ and for every i such that $i \in \operatorname{dom} G$ holds $G(i) = F(f(i))$ holds $\sum F = \sum G$.

(9) For every permutation f of $\operatorname{dom} F$ such that $G = F \cdot f$ holds $\sum F = \sum G$.

Let us consider V. A subset of the vectors of V is called a finite subset of V if:

it is finite.

One can prove the following proposition

(10) A is a finite subset of V if and only if A is finite.

In the sequel S, T will be finite subsets of V. Let us consider V, S, T. Then $S \cup T$ is a finite subset of V. Then $S \cap T$ is a finite subset of V. Then $S \setminus T$ is a finite subset of V. Then $S \dot{-} T$ is a finite subset of V.

Let us consider V. The functor 0_V yielding a finite subset of V, is defined by:

$0_V = \emptyset$.

One can prove the following proposition

(11) $0_V = \emptyset$.

Let us consider V, T. The functor $\sum T$ yields a vector of V and is defined as follows:

there exists F such that $\operatorname{rng} F = T$ and F is one-to-one and $\sum T = \sum F$.

One can prove the following propositions:

(12) There exists F such that $\operatorname{rng} F = T$ and F is one-to-one and $\sum T = \sum F$.

(13) If $\operatorname{rng} F = T$ and F is one-to-one and $v = \sum F$, then $v = \sum T$.

Let us consider V, v. Then $\{v\}$ is a finite subset of V.

Let us consider V, v_1, v_2. Then $\{v_1, v_2\}$ is a finite subset of V.

Let us consider V, v_1, v_2, v_3. Then $\{v_1, v_2, v_3\}$ is a finite subset of V.

One can prove the following propositions:

(14) $\sum(0_V) = 0_V$.

(15) $\sum\{v\} = v$.

(16) If $v_1 \neq v_2$, then $\sum\{v_1, v_2\} = v_1 + v_2$.

(17) If $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_1 \neq v_3$, then $\sum\{v_1, v_2, v_3\} = (v_1 + v_2) + v_3$.

(18) If T misses S, then $\sum(T \cup S) = \sum T + \sum S$.

(19) $\sum(T \cup S) = (\sum T + \sum S) - \sum(T \cap S)$.

(20) $\sum(T \cap S) = (\sum T + \sum S) - \sum(T \cup S)$.

(21) $\sum(T \setminus S) = \sum(T \cup S) - \sum S$.

(22) $\sum(T \setminus S) = \sum T - \sum(T \cap S)$.

(23) $\sum(T \dot{-} S) = \sum(T \cup S) - \sum(T \cap S)$.

(24) $\sum(T \dot{-} S) = \sum(T \setminus S) + \sum(S \setminus T)$.

Let us consider V. An element of $\mathbb{R}^{\text{the vectors of } V}$ is called a linear combination of V if:

there exists T such that for every v such that $v \notin T$ holds $\operatorname{it}(v) = 0$.

In the sequel K, L, L_1, L_2, L_3 will be linear combinations of V. Next we state a proposition

(25) There exists T such that for every v such that $v \notin T$ holds $L(v) = 0$.

In the sequel E denotes an element of $\mathbb{R}^{\text{the vectors of } V}$. We now state a proposition

(26) If there exists T such that for every v such that $v \notin T$ holds $E(v) = 0$, then E is a linear combination of V.

Let us consider V, L. The functor $\operatorname{support} L$ yields a finite subset of V and is defined as follows:

$\operatorname{support} L = \{v : L(v) \neq 0\}$.

We now state two propositions:

(27) $\operatorname{support} L = \{v : L(v) \neq 0\}$.

(28) $L(v) = 0$ if and only if $v \notin \operatorname{support} L$.

Let us consider V. The functor $\mathbf{0}_{\mathrm{LC}_V}$ yields a linear combination of V and is defined as follows:

$\operatorname{support} \mathbf{0}_{\mathrm{LC}_V} = \emptyset$.

The following propositions are true:

(29) $L = \mathbf{0}_{\mathrm{LC}_V}$ if and only if $\operatorname{support} L = \emptyset$.

(30) $\mathbf{0}_{\mathrm{LC}_V}(v) = 0$.

Let us consider V, A. A linear combination of V is said to be a linear combination of A if:

support it $\subseteq A$.

One can prove the following proposition

(31) If support $L \subseteq A$, then L is a linear combination of A.

In the sequel l is a linear combination of A. The following propositions are true:

(32) support $l \subseteq A$.

(33) If $A \subseteq B$, then l is a linear combination of B.

(34) $\mathbf{0}_{\mathrm{LC}_V}$ is a linear combination of A.

(35) For every linear combination l of $\emptyset_{\text{the vectors of } V}$ holds $l = \mathbf{0}_{\mathrm{LC}_V}$.

(36) L is a linear combination of support L.

Let us consider V, F, f. The functor $f \cdot F$ yields a finite sequence of elements of the vectors of V and is defined as follows:

$\mathrm{len}(f \cdot F) = \mathrm{len}\, F$ and for every i such that $i \in \mathrm{dom}(f \cdot F)$ holds $(f \cdot F)(i) = f(@(\pi_i F)) \cdot @(\pi_i F)$.

Next we state several propositions:

(37) $\mathrm{len}(f \cdot F) = \mathrm{len}\, F$.

(38) For every i such that $i \in \mathrm{dom}(f \cdot F)$ holds $(f \cdot F)(i) = f(@(\pi_i F)) \cdot @(\pi_i F)$.

(39) If $\mathrm{len}\, G = \mathrm{len}\, F$ and for every i such that $i \in \mathrm{dom}\, G$ holds $G(i) = f(@(\pi_i F)) \cdot @(\pi_i F)$, then $G = f \cdot F$.

(40) If $i \in \mathrm{dom}\, F$ and $v = F(i)$, then $(f \cdot F)(i) = f(v) \cdot v$.

(41) $f \cdot \varepsilon_{\text{the vectors of } V} = \varepsilon_{\text{the vectors of } V}$.

(42) $f \cdot \langle v \rangle = \langle f(v) \cdot v \rangle$.

(43) $f \cdot \langle v_1, v_2 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2 \rangle$.

(44) $f \cdot \langle v_1, v_2, v_3 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2, f(v_3) \cdot v_3 \rangle$.

Let us consider V, L. The functor $\sum L$ yields a vector of V and is defined by:

there exists F such that F is one-to-one and $\mathrm{rng}\, F = \mathrm{support}\, L$ and $\sum L = \sum (L \cdot F)$.

The following propositions are true:

(45) There exists F such that F is one-to-one and $\mathrm{rng}\, F = \mathrm{support}\, L$ and $\sum L = \sum (L \cdot F)$.

(46) If F is one-to-one and $\mathrm{rng}\, F = \mathrm{support}\, L$ and $u = \sum (L \cdot F)$, then $u = \sum L$.

(47) $A \neq \emptyset$ and A is linearly closed if and only if for every l holds $\sum l \in A$.

(48) $\sum \mathbf{0}_{\mathrm{LC}_V} = 0_V$.

(49) For every linear combination l of $\emptyset_{\text{the vectors of } V}$ holds $\sum l = 0_V$.

(50) For every linear combination l of $\{v\}$ holds $\sum l = l(v) \cdot v$.

(51) If $v_1 \neq v_2$, then for every linear combination l of $\{v_1, v_2\}$ holds $\sum l = l(v_1) \cdot v_1 + l(v_2) \cdot v_2$.

(52) If $\operatorname{support} L = \emptyset$, then $\sum L = 0_V$.

(53) If $\operatorname{support} L = \{v\}$, then $\sum L = L(v) \cdot v$.

(54) If $\operatorname{support} L = \{v_1, v_2\}$ and $v_1 \neq v_2$, then $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

Let us consider V, L_1, L_2. Let us note that one can characterize the predicate $L_1 = L_2$ by the following (equivalent) condition: for every v holds $L_1(v) = L_2(v)$.

One can prove the following proposition

(55) If for every v holds $L_1(v) = L_2(v)$, then $L_1 = L_2$.

Let us consider V, L_1, L_2. The functor $L_1 + L_2$ yields a linear combination of V and is defined as follows:

for every v holds $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

The following propositions are true:

(56) If for every v holds $L(v) = L_1(v) + L_2(v)$, then $L = L_1 + L_2$.

(57) $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

(58) $\operatorname{support}(L_1 + L_2) \subseteq \operatorname{support} L_1 \cup \operatorname{support} L_2$.

(59) If L_1 is a linear combination of A and L_2 is a linear combination of A, then $L_1 + L_2$ is a linear combination of A.

(60) $L_1 + L_2 = L_2 + L_1$.

(61) $L_1 + (L_2 + L_3) = (L_1 + L_2) + L_3$.

(62) $L + \mathbf{0}_{\mathrm{LC}_V} = L$ and $\mathbf{0}_{\mathrm{LC}_V} + L = L$.

Let us consider V, a, L. The functor $a \cdot L$ yielding a linear combination of V, is defined by:

for every v holds $(a \cdot L)(v) = a \cdot L(v)$.

The following propositions are true:

(63) If for every v holds $K(v) = a \cdot L(v)$, then $K = a \cdot L$.

(64) $(a \cdot L)(v) = a \cdot L(v)$.

(65) If $a \neq 0$, then $\operatorname{support}(a \cdot L) = \operatorname{support} L$.

(66) $0 \cdot L = \mathbf{0}_{\mathrm{LC}_V}$.

(67) If L is a linear combination of A, then $a \cdot L$ is a linear combination of A.

(68) $(a + b) \cdot L = a \cdot L + b \cdot L$.

(69) $a \cdot (L_1 + L_2) = a \cdot L_1 + a \cdot L_2$.

(70) $a \cdot (b \cdot L) = (a \cdot b) \cdot L$.

(71) $1 \cdot L = L$.

Let us consider V, L. The functor $-L$ yielding a linear combination of V, is defined as follows:

$-L = (-1) \cdot L$.

Next we state several propositions:

(72) $-L = (-1) \cdot L$.

(73) $(-L)(v) = -L(v)$.

(74) If $L_1 + L_2 = \mathbf{0}_{\mathrm{LC}_V}$, then $L_2 = -L_1$.

(75) $\mathrm{support}(-L) = \mathrm{support}\, L$.

(76) If L is a linear combination of A, then $-L$ is a linear combination of A.

(77) $-(-L) = L$.

Let us consider V, L_1, L_2. The functor $L_1 - L_2$ yields a linear combination of V and is defined by:

$L_1 - L_2 = L_1 + (-L_2)$.

The following propositions are true:

(78) $L_1 - L_2 = L_1 + (-L_2)$.

(79) $(L_1 - L_2)(v) = L_1(v) - L_2(v)$.

(80) $\mathrm{support}(L_1 - L_2) \subseteq \mathrm{support}\, L_1 \cup \mathrm{support}\, L_2$.

(81) If L_1 is a linear combination of A and L_2 is a linear combination of A, then $L_1 - L_2$ is a linear combination of A.

(82) $L - L = \mathbf{0}_{\mathrm{LC}_V}$.

Let us consider V. The functor LC_V yields a non-empty set and is defined by:

$x \in \mathrm{LC}_V$ if and only if x is a linear combination of V.

In the sequel D denotes a non-empty set and e, e_1, e_2 denote elements of LC_V. The following propositions are true:

(83) If for every x holds $x \in D$ if and only if x is a linear combination of V, then $D = \mathrm{LC}_V$.

(84) $L \in \mathrm{LC}_V$.

Let us consider V, e. The functor $@e$ yields a linear combination of V and is defined by:

$@e = e$.

The following proposition is true

(85) $@e = e$.

Let us consider V, L. The functor $@L$ yields an element of LC_V and is defined as follows:

$@L = L$.

Next we state a proposition

(86) $@L = L$.

Let us consider V. The functor $+_{\mathrm{LC}_V}$ yields a binary operation on LC_V and is defined by:

for all e_1, e_2 holds $+_{\mathrm{LC}_V}(e_1, e_2) = @e_1 + @e_2$.

In the sequel o is a binary operation on LC_V. Next we state two propositions:

(87) If for all e_1, e_2 holds $o(e_1, e_2) = @e_1 + @e_2$, then $o = +_{\mathrm{LC}_V}$.

(88) $+_{\mathrm{LC}_V}(e_1, e_2) = @e_1 + @e_2$.

Let us consider V. The functor $\cdot_{\mathrm{LC}_V}$ yields a function from $[\!:\mathbb{R}, \mathrm{LC}_V:\!]$ into LC_V and is defined as follows:

for all a, e holds $\cdot_{\mathrm{LC}_V}(\langle a, e \rangle) = a \cdot @e$.

In the sequel g denotes a function from $[\!:\mathbb{R}, \mathrm{LC}_V:\!]$ into LC_V. We now state two propositions:

(89) If for all a, e holds $g(\langle a, e\rangle) = a \cdot @e$, then $g = \cdot_{\mathrm{LC}_V}$.

(90) $\cdot_{\mathrm{LC}_V}(\langle a, e\rangle) = a \cdot @e$.

Let us consider V. The functor $\mathbb{LC}_V$ yielding a real linear space, is defined as follows:

$\mathbb{LC}_V = \langle \mathrm{LC}_V, @\mathbf{0}_{\mathrm{LC}_V}, +_{\mathrm{LC}_V}, \cdot_{\mathrm{LC}_V}\rangle$.

Next we state several propositions:

(91) $\mathbb{LC}_V = \langle \mathrm{LC}_V, @\mathbf{0}_{\mathrm{LC}_V}, +_{\mathrm{LC}_V}, \cdot_{\mathrm{LC}_V}\rangle$.

(92) The vectors of $\mathbb{LC}_V = \mathrm{LC}_V$.

(93) The zero of $\mathbb{LC}_V = \mathbf{0}_{\mathrm{LC}_V}$.

(94) The addition of $\mathbb{LC}_V = +_{\mathrm{LC}_V}$.

(95) The multiplication$_1$ of $\mathbb{LC}_V = \cdot_{\mathrm{LC}_V}$.

(96) ${L_1}^{\mathbb{LC}_V} + {L_2}^{\mathbb{LC}_V} = L_1 + L_2$.

(97) $a \cdot L^{\mathbb{LC}_V} = a \cdot L$.

(98) $-L^{\mathbb{LC}_V} = -L$.

(99) ${L_1}^{\mathbb{LC}_V} - {L_2}^{\mathbb{LC}_V} = L_1 - L_2$.

Let us consider V, A. The functor $\mathbb{LC}_A$ yielding a subspace of $\mathbb{LC}_V$, is defined by:

the vectors of $\mathbb{LC}_A = \{l\}$.

In the sequel W denotes a subspace of $\mathbb{LC}_V$. Next we state two propositions:

(100) If the vectors of $W = \{l\}$, then $W = \mathbb{LC}_A$.

(101) The vectors of $\mathbb{LC}_A = \{l\}$.

We now state several propositions:

(102) $X \setminus Y$ misses $Y \setminus X$.

(103) If $k < n$, then $n - 1$ is a natural number.

(104) $-1 \neq 0$.

(105) $(-1) \cdot r = -r$.

(106) $r - 1 < r$.

(107) If X is finite and Y is finite, then $X \dot{-} Y$ is finite.

(108) For every function f such that $f^{-1}\, X = f^{-1}\, Y$ and $X \subseteq \mathrm{rng}\, f$ and $Y \subseteq \mathrm{rng}\, f$ holds $X = Y$.

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