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Convergent Real Sequences. Upper and Lower Bound of Sets of Real Numbers

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Summary. The article contains theorems about convergent sequences and the limit of sequences occurring in [3] such as Bolzano-Weirrstrass theorem, Cauchy theorem and others. Bounded sets of real numbers and lower and upper bound of subset of real numbers are defined.

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The papers [7], [2], [5], [3], [1], [4], [8], and [6] provide the notation and terminology for this paper. For simplicity we follow a convention: n, k, m will denote natural numbers, r, r_1, p, g, g_1, g_2, s will denote real numbers, seq, seq_1 will denote sequences of real numbers, $Nseq$ will denote an increasing sequence of naturals, and X, Y will denote subsets of $\mathbb{R}$. One can prove the following propositions:

(1) If $0 < r_1$ and $r_1 \leq r$ and $0 < g$, then $\frac{g}{r} \leq \frac{g}{r_1}$.

(2) If $r < p$, then $0 < p - r$.

(3) $r - (r - s) = s$ and $r + (s - r) = s$ and $(r + s) - r = s$.

(4) If $0 < s$, then $0 < \frac{s}{3}$.

(5) $(\frac{s}{3} + \frac{s}{3}) + \frac{s}{3} = s$.

(6) If $0 < g$ and $0 < r$ and $g \leq g_1$ and $r < r_1$, then $g \cdot r < g_1 \cdot r_1$ and $r \cdot g < r_1 \cdot g_1$.

(7) If $0 < g$ and $0 < r$ and $g \leq g_1$ and $r \leq r_1$, then $g \cdot r \leq g_1 \cdot r_1$ and $r \cdot g \leq r_1 \cdot g_1$.

(8) Given X, Y. Then if there exists r such that $r \in X$ and there exists r such that $r \in Y$ and for all r, p such that $r \in X$ and $p \in Y$ holds $r < p$, then there exists g such that for all r, p such that $r \in X$ and $p \in Y$ holds $r \leq g$ and $g \leq p$.

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(9) If $0 < p$ and there exists r such that $r \in X$ and for every r such that $r \in X$ holds $r + p \in X$, then for every g there exists r such that $r \in X$ and $g < r$.

(10) For every r there exists n such that $r < n$.

We now define two new predicates. Let us consider X. Let us assume that there exists r such that $r \in X$. We say that X is upper bounded if and only if:

there exists p such that for every r such that $r \in X$ holds $r \leq p$.

We say that X is lower bounded if and only if:

there exists p such that for every r such that $r \in X$ holds $p \leq r$.

Let us consider X. Let us assume that there exists r such that $r \in X$. We say that X is bounded if and only if:

X is lower bounded and X is upper bounded.

We now state several propositions:

(11) If there exists r such that $r \in X$, then X is upper bounded if and only if there exists p such that for every r such that $r \in X$ holds $r \leq p$.

(12) If there exists r such that $r \in X$, then X is lower bounded if and only if there exists p such that for every r such that $r \in X$ holds $p \leq r$.

(13) If there exists r such that $r \in X$, then X is bounded if and only if X is upper bounded and X is lower bounded.

(14) If there exists r such that $r \in X$, then X is bounded if and only if there exists s such that $0 < s$ and for every r such that $r \in X$ holds $|r| < s$.

(15) If $X = \{r\}$, then X is bounded.

(16) If there exists r such that $r \in X$ and X is upper bounded, then there exists g such that for every r such that $r \in X$ holds $r \leq g$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $g - s < r$.

(17) Suppose that
   (i) for every r such that $r \in X$ holds $r \leq g_1$,
   (ii) for every s such that $0 < s$ there exists r such that $r \in X$ and $g_1 - s < r$,
   (iii) for every r such that $r \in X$ holds $r \leq g_2$,
   (iv) for every s such that $0 < s$ there exists r such that $r \in X$ and $g_2 - s < r$.

   Then $g_1 = g_2$.

(18) If there exists r such that $r \in X$ and X is lower bounded, then there exists g such that for every r such that $r \in X$ holds $g \leq r$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g + s$.

(19) Suppose that
   (i) for every r such that $r \in X$ holds $g_1 \leq r$,
   (ii) for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g_1 + s$,
   (iii) for every r such that $r \in X$ holds $g_2 \leq r$,
   (iv) for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g_2 + s$.

   Then $g_1 = g_2$.

Let us consider X. Let us assume that there exists r such that $r \in X$ and X is upper bounded. The functor $\sup X$ yielding a real number, is defined as follows:

for every r such that $r \in X$ holds $r \leq \sup X$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $(\sup X) - s < r$.

Let us consider X. Let us assume that there exists r such that $r \in X$ and X is lower bounded. The functor $\inf X$ yields a real number and is defined by:

for every r such that $r \in X$ holds $\inf X \leq r$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $r < (\inf X) + s$.

One can prove the following propositions:

(20) If there exists r such that $r \in X$ and X is upper bounded, then $\sup X = g$ if and only if for every r such that $r \in X$ holds $r \leq g$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $g - s < r$.

(21) If there exists r such that $r \in X$ and X is lower bounded, then $\inf X = g$ if and only if for every r such that $r \in X$ holds $g \leq r$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g + s$.

(22) If $X = \{r\}$, then $\inf X = r$ and $\sup X = r$.

(23) If $X = \{r\}$, then $\inf X = \sup X$.

(24) If X is bounded and there exists r such that $r \in X$, then $\inf X \leq \sup X$.

(25) If X is bounded and there exists r such that $r \in X$, then there exist r, p such that $r \in X$ and $p \in X$ and $p \neq r$ if and only if $\inf X < \sup X$.

The scheme *SepNat* concerns a unary predicate $\mathcal{P}$, and states that:

there exists a X being sets of natural numbers such that for every n holds $n \in X$ if and only if $\mathcal{P}[n]$

for all values of the parameter.

We now state a number of propositions:

(26) If *seq* is convergent, then $|seq|$ is convergent.

(27) If *seq* is convergent, then $\lim |seq| = |\lim seq|$.

(28) If $|seq|$ is convergent and $\lim |seq| = 0$, then *seq* is convergent and $\lim seq = 0$.

(29) If seq_1 is a subsequence of *seq* and *seq* is convergent, then seq_1 is convergent.

(30) If seq_1 is a subsequence of *seq* and *seq* is convergent, then $\lim seq_1 = \lim seq$.

(31) If *seq* is convergent and there exists k such that for every n such that $k \leq n$ holds $seq_1(n) = seq(n)$, then seq_1 is convergent.

(32) If *seq* is convergent and there exists k such that for every n such that $k \leq n$ holds $seq_1(n) = seq(n)$, then $\lim seq = \lim seq_1$.

(33) If *seq* is convergent, then $seq\frown k$ is convergent and $\lim(seq\frown k) = \lim seq$.

(34) If *seq* is convergent and there exists k such that $seq_1 = seq \frown k$, then seq_1 is convergent and $\lim seq_1 = \lim seq$.

(35) If *seq* is convergent and there exists k such that $seq = seq_1 \frown k$, then seq_1 is convergent.

(36) If *seq* is convergent and there exists k such that $seq = seq_1 \frown k$, then $\lim seq_1 = \lim seq$.

(37) If seq is convergent and $\lim seq \neq 0$, then there exists k such that $seq \frown k$ is non-zero.

(38) If seq is convergent and $\lim seq \neq 0$, then there exists seq_1 such that seq_1 is a subsequence of seq and seq_1 is non-zero.

(39) If seq is constant, then seq is convergent.

(40) If seq is constant and $r \in \operatorname{rng} seq$ or seq is constant and there exists n such that $seq(n) = r$, then $\lim seq = r$.

(41) If seq is constant, then for every n holds $\lim seq = seq(n)$.

(42) If seq is convergent and $\lim seq \neq 0$, then for every seq_1 such that seq_1 is a subsequence of seq and seq_1 is non-zero holds $\lim {seq_1}^{-1} = (\lim seq)^{-1}$.

(43) For all r, seq such that $0 < r$ and for every n holds $seq(n) = \frac{1}{n+r}$ holds seq is convergent.

(44) For all r, seq such that $0 < r$ and for every n holds $seq(n) = \frac{1}{n+r}$ holds $\lim seq = 0$.

(45) If for every n holds $seq(n) = \frac{1}{n+1}$, then seq is convergent and $\lim seq = 0$.

(46) If $0 < r$ and for every n holds $seq(n) = \frac{g}{n+r}$, then seq is convergent and $\lim seq = 0$.

(47) For all r, seq such that $0 < r$ and for every n holds $seq(n) = \frac{1}{n \cdot n+r}$ holds seq is convergent.

(48) For all r, seq such that $0 < r$ and for every n holds $seq(n) = \frac{1}{n \cdot n+r}$ holds $\lim seq = 0$.

(49) If for every n holds $seq(n) = \frac{1}{n \cdot n+1}$, then seq is convergent and $\lim seq = 0$.

(50) If $0 < r$ and for every n holds $seq(n) = \frac{g}{n \cdot n+r}$, then seq is convergent and $\lim seq = 0$.

(51) If seq is non-decreasing and seq is upper bounded, then seq is convergent.

(52) If seq is non-increasing and seq is lower bounded, then seq is convergent.

(53) If seq is monotone and seq is bounded, then seq is convergent.

(54) If seq is upper bounded and seq is non-decreasing, then for every n holds $seq(n) \leq \lim seq$.

(55) If seq is lower bounded and seq is non-increasing, then for every n holds $\lim seq \leq seq(n)$.

(56) For every seq there exists $Nseq$ such that $seq \cdot Nseq$ is monotone.

(57) If seq is bounded, then there exists seq_1 such that seq_1 is a subsequence of seq and seq_1 is convergent.

(58) seq is convergent if and only if for every s such that $0 < s$ there exists n such that for every m such that $n \leq m$ holds $|seq(m) - seq(n)| < s$.

(59) Suppose seq is constant and seq_1 is convergent. Then $\lim(seq + seq_1) = seq(0) + \lim seq_1$ and $\lim(seq - seq_1) = seq(0) - \lim seq_1$ and $\lim(seq_1 -$

$seq) = \lim seq_1 - seq(0)$ and $\lim(seq \cdot seq_1) = seq(0) \cdot (\lim seq_1)$.

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