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Zermelo's Theorem [1]

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Summary. The article contains direct proof of Zermelo's theorem about the existence of a well ordering for any set and the lemma the proof depends on.

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The articles [4], [3], [5], [2], and [1] provide the notation and terminology for this paper. For simplicity we follow the rules: a, x, y will be arbitrary, B, D, N, X, Y will denote sets, R, S, T will denote relations, F will denote a function, and W will denote a relation. We now state several propositions:

(1) $x \in \text{field}\, R$ if and only if there exists y such that $\langle x, y\rangle \in R$ or $\langle y, x\rangle \in R$.

(2) $R \cup S$ is a relation.

(3) If $X \neq \emptyset$ and $Y \neq \emptyset$ and $W = [\!: X, Y :\!]$, then $\text{field}\, W = X \cup Y$.

(4) If $y = R$, then y is a relation.

(5) For all a, T holds $x \in T{-}\text{Seg}(a)$ if and only if $x \neq a$ and $\langle x, a\rangle \in T$.

In the article we present several logical schemes. The scheme *R_Separation* deals with a set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists B such that for every relation R holds $R \in B$ if and only if $R \in \mathcal{A}$ and $\mathcal{P}[R]$

for all values of the parameters.

The scheme *S_Separation* deals with a set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists B such that for every set X holds $X \in B$ if and only if $X \in \mathcal{A}$ and $\mathcal{P}[X]$

for all values of the parameters.

The following four propositions are true:

(6) For all x, y, W such that $x \in \text{field}\, W$ and $y \in \text{field}\, W$ and W is well ordering relation holds if $x \notin W{-}\text{Seg}(y)$, then $\langle y, x\rangle \in W$.

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(7) For all x, y, W such that $x \in \operatorname{field} W$ and $y \in \operatorname{field} W$ and W is well ordering relation holds if $x \in W{-}\mathrm{Seg}(y)$, then $\langle y, x\rangle \notin W$.

(8) Given F, D. Suppose for every X such that $X \in D$ holds $F(X) \notin X$ and $F(X) \in \bigcup D$. Then there exists R such that $\operatorname{field} R \subseteq \bigcup D$ and R is well ordering relation and $\operatorname{field} R \notin D$ and for every y such that $y \in \operatorname{field} R$ holds $R{-}\mathrm{Seg}(y) \in D$ and $F(R{-}\mathrm{Seg}(y)) = y$.

(9) For every N there exists R such that R is well ordering relation and $\operatorname{field} R = N$.

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