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A Construction of Analytical Projective Space

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Summary. The collinearity structure denoted by ProjectiveSpace(V) is correlated with a given vector space V (over the field of Reals). It is a formalization of the standard construction of a projective space, where points are interpreted as equivalence classes of the relation of proportionality considered in the set of all non-zero vectors. Then the relation of collinearity corresponds to the relation of linear dependence of vectors. Several facts concerning vectors are proved, which correspond in this language to some classical axioms of projective geometry.

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The notation and terminology used here are introduced in the following articles: [7], [8], [6], [2], [3], [4], [5], [1], and [9]. We adopt the following rules: V is a real linear space, p, q, r, u, v, w, y, u_1, v_1, w_1 are vectors of V, and a, b, c, a_1, b_1, c_1, a_2, b_2, c_2 are real numbers. Let us consider V, p. We say that p is a proper vector if and only if:

$p \neq 0_V$.

The following proposition is true

(1) p is a proper vector if and only if $p \neq 0_V$.

Let us consider V, p, q. We say that p and q are proportional if and only if: there exist a, b such that $a \cdot p = b \cdot q$ and $a \neq 0$ and $b \neq 0$.

One can prove the following propositions:

(2) p and q are proportional if and only if there exist a, b such that $a \cdot p = b \cdot q$ and $a \neq 0$ and $b \neq 0$.

(3) p and p are proportional.

(4) If p and q are proportional, then q and p are proportional.

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(5) p and q are proportional if and only if there exists a such that $a \neq 0$ and $p = a \cdot q$.

(6) If p and u are proportional and u and q are proportional, then p and q are proportional.

(7) p and 0_V are proportional if and only if $p = 0_V$.

Let us consider V, u, v, w. We say that u, v and w are lineary dependent if and only if:

there exist a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ but $a \neq 0$ or $b \neq 0$ or $c \neq 0$.

We now state a number of propositions:

(8) u, v and w are lineary dependent if and only if there exist a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ but $a \neq 0$ or $b \neq 0$ or $c \neq 0$.

(9) If u and u_1 are proportional and v and v_1 are proportional and w and w_1 are proportional and u, v and w are lineary dependent, then u_1, v_1 and w_1 are lineary dependent.

(10) If u, v and w are lineary dependent, then u, w and v are lineary dependent and v, u and w are lineary dependent and w, v and u are lineary dependent and w, u and v are lineary dependent and v, w and u are lineary dependent.

(11) If v is a proper vector and w is a proper vector and v and w are not proportional, then v, w and u are lineary dependent if and only if there exist a, b such that $u = a \cdot v + b \cdot w$.

(12) If p and q are not proportional and $a_1 \cdot p + b_1 \cdot q = a_2 \cdot p + b_2 \cdot q$ and p is a proper vector and q is a proper vector, then $a_1 = a_2$ and $b_1 = b_2$.

(13) If u, v and w are not lineary dependent and $(a_1 \cdot u + b_1 \cdot v) + c_1 \cdot w = (a_2 \cdot u + b_2 \cdot v) + c_2 \cdot w$, then $a_1 = a_2$ and $b_1 = b_2$ and $c_1 = c_2$.

(14) Suppose p and q are not proportional and $u = a_1 \cdot p + b_1 \cdot q$ and $v = a_2 \cdot p + b_2 \cdot q$ and $a_1 \cdot b_2 - a_2 \cdot b_1 = 0$ and p is a proper vector and q is a proper vector. Then u and v are proportional or $u = 0_V$ or $v = 0_V$.

(15) If $u = 0_V$ or $v = 0_V$ or $w = 0_V$, then u, v and w are lineary dependent.

(16) If u and v are proportional or w and u are proportional or v and w are proportional, then w, u and v are lineary dependent.

(17) If u, v and w are not lineary dependent, then u is a proper vector and v is a proper vector and w is a proper vector and u and v are not proportional and v and w are not proportional and w and u are not proportional.

(18) If $p + q = 0_V$, then p and q are proportional.

(19) If p and q are not proportional and p, q and u are lineary dependent and p, q and v are lineary dependent and p, q and w are lineary dependent and p is a proper vector and q is a proper vector, then u, v and w are lineary dependent.

(20) If u, v and w are not lineary dependent and u, v and p are lineary dependent and v, w and q are lineary dependent, then there exists y such

that u, w and y are lineary dependent and p, q and y are lineary dependent and y is a proper vector.

(21) If p and q are not proportional and p is a proper vector and q is a proper vector, then for every u, v there exists y such that y is a proper vector and u, v and y are lineary dependent and u and y are not proportional and v and y are not proportional.

(22) If p, q and r are not lineary dependent, then for all u, v such that u is a proper vector and v is a proper vector and u and v are not proportional there exists y such that y is a proper vector and u, v and y are not lineary dependent.

(23) Suppose u, v and q are lineary dependent and w, y and q are lineary dependent and u, w and p are lineary dependent and v, y and p are lineary dependent and u, y and r are lineary dependent and v, w and r are lineary dependent and p, q and r are lineary dependent and p is a proper vector and q is a proper vector and r is a proper vector. Then u, v and y are lineary dependent or u, v and w are lineary dependent or u, w and y are lineary dependent or v, w and y are lineary dependent.

In the sequel x, y, z are arbitrary and X denotes a set. Let us consider V. The proper vectors ofV yields a set and is defined as follows:

for an arbitrary u holds $u \in$ the proper vectors ofV if and only if $u \neq 0_V$ and u is a vector of V.

Next we state three propositions:

(24) For every X holds $X =$ the proper vectors ofV if and only if for an arbitrary u holds $u \in X$ if and only if $u \neq 0_V$ and u is a vector of V.

(25) For an arbitrary u such that $u \in$ the proper vectors ofV holds u is a vector of V.

(26) For every u holds $u \in$ the proper vectors ofV if and only if u is a proper vector.

Let us consider V. The proportionality in V yields an equivalence relation of the proper vectors ofV and is defined as follows:

for all x, y holds $\langle x, y\rangle \in$ the proportionality in V if and only if $x \in$ the proper vectors ofV and $y \in$ the proper vectors ofV and there exist vectors u, v of V such that $x = u$ and $y = v$ and u and v are proportional.

We now state three propositions:

(27) For every equivalence relation R of the proper vectors ofV holds $R =$ the proportionality in V if and only if for all x, y holds $\langle x, y\rangle \in R$ if and only if $x \in$ the proper vectors ofV and $y \in$ the proper vectors ofV and there exist vectors u, v of V such that $x = u$ and $y = v$ and u and v are proportional.

(28) If $\langle x, y\rangle \in$ the proportionality in V, then x is a vector of V and y is a vector of V.

(29) $\langle u, v\rangle \in$ the proportionality in V if and only if u is a proper vector and v is a proper vector and u and v are proportional.

Let us consider V, v. Let us assume that v is a proper vector. The direction of v yields a subset of the proper vectors ofV and is defined by:

the direction of $v = [v]_{\text{the proportionality in } V}$.

We now state the proposition

(30) If v is a proper vector, then the direction of $v = [v]_{\text{the proportionality in } V}$.

Let us consider V. The projective points overV yields a set and is defined as follows:

there exists a family Y of subsets of the proper vectors ofV such that $Y =$ Classes(the proportionality in V) and the projective points overV = Y.

The following proposition is true

(31) For every X holds $X =$ the projective points overV if and only if there exists a family Y of subsets of the proper vectors ofV such that $Y =$ Classes(the proportionality in V) and $X = Y$.

A real linear space is said to be a non-trivial real linear space if:

there exists a vector u of it such that $u \neq 0_{\text{it}}$.

The following two propositions are true:

(32) For every real linear space V holds V is a non-trivial real linear space if and only if there exists a vector u of V such that $u \neq 0_V$.

(33) For every real linear space V holds V is a non-trivial real linear space if and only if there exists u such that $u \in$ the proper vectors ofV.

We follow the rules: V will denote a non-trivial real linear space, p, q, r, u, v, w will denote vectors of V, and y will be arbitrary. Let us consider V. Then the proper vectors ofV is a non-empty set.

Let us consider V. Then the projective points overV is a non-empty set.

Next we state two propositions:

(34) If p is a proper vector, then the direction of p is an element of the projective points overV.

(35) If p is a proper vector and q is a proper vector, then the direction of $p =$ the direction of q if and only if p and q are proportional.

Let us consider V. The projective collinearity overV yielding a ternary relation on the projective points overV is defined by:

for arbitrary x, y, z holds $\langle x, y, z \rangle \in$ the projective collinearity overV if and only if there exist p, q, r such that $x =$ the direction of p and $y =$ the direction of q and $z =$ the direction of r and p is a proper vector and q is a proper vector and r is a proper vector and p, q and r are lineary dependent.

We now state the proposition

(36) Let R be a ternary relation on the projective points overV. Then $R =$ the projective collinearity overV if and only if for arbitrary x, y, z holds $\langle x, y, z \rangle \in R$ if and only if there exist p, q, r such that $x =$ the direction of p and $y =$ the direction of q and $z =$ the direction of r and p is a proper vector and q is a proper vector and r is a proper vector and p, q and r are lineary dependent.

Let us consider V. The projective space over V yields a collinearity structure and is defined by:

the projective space over V = $\langle$the projective points overV, the projective collinearity overV$\rangle$.

In the sequel CS will be a collinearity structure. One can prove the following propositions:

(37) For every CS holds CS = the projective space over V if and only if CS = $\langle$the projective points overV, the projective collinearity overV$\rangle$.

(38) The projective space over V = $\langle$the projective points overV, the projective collinearity overV$\rangle$.

(39) For every V holds the points of the projective space over V = the projective points overV and the collinearity relation of the projective space over V = the projective collinearity overV.

(40) If $\langle x, y, z \rangle \in$ the collinearity relation of the projective space over V, then there exist p, q, r such that x = the direction of p and y = the direction of q and z = the direction of r and p is a proper vector and q is a proper vector and r is a proper vector and p, q and r are lineary dependent.

(41) If u is a proper vector and v is a proper vector and w is a proper vector, then $\langle$ the direction of u, the direction of v, the direction of $w\rangle \in$ the collinearity relation of the projective space over V if and only if u, v and w are lineary dependent.

(42) x is an element of the points of the projective space over V if and only if there exists u such that u is a proper vector and x = the direction of u.

(43) For every real linear space V and for every vector v of V such that v is a proper vector for every subset X of the proper vectors ofV holds X = the direction of v if and only if $X = [v]_{\text{the proportionality in } V}$.

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