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The Collinearity Structure

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Summary. The text includes basic axioms and theorems concerning the collinearity structure based on Wanda Szmielew [1], pp. 18-20. Collinearity is defined as a relation on Cartesian product $[:S,S,S:]$ of set S. The basic text is preceeded with a few auxiliary theorems (e.g: ternary relation). Then come the two basic axioms of the collinearity structure: A1.1.1 and A1.1.2 and a few theorems. Another axiom: Aks dim, which states that there exist at least 3 non-collinear points, excludes the trivial structures (i.e. pairs $\langle S,[:S,S,S:]\rangle$). Following it the notion of a line is included and several additional theorems are appended.

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The articles [3], and [2] provide the notation and terminology for this paper. In the sequel R, X will denote sets. Let us consider X. A set is said to be a ternary relation on X if:

it $\subseteq [: X, X, X :]$.

Next we state two propositions:

(1) R is a ternary relation on X if and only if $R \subseteq [: X, X, X :]$.

(2) $X = \emptyset$ or there exists arbitrary a such that $\{a\} = X$ or there exist arbitrary a, b such that $a \neq b$ and $a \in X$ and $b \in X$.

We consider collinearity structures which are systems

$\langle$points, a collinearity relation$\rangle$,

where the points constitute a non-empty set and the collinearity relation is a ternary relation on the points. In the sequel CS is a collinearity structure. Let us consider CS. A point of CS is an element of the points of CS.

In the sequel a, b, c denote points of CS. Let us consider CS, a, b, c. We say that a, b and c are collinear if and only if:

$\langle a,b,c\rangle \in$ the collinearity relation of CS.

The following proposition is true

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(5)[2] a, b and c are collinear if and only if $\langle a,b,c\rangle \in$ the collinearity relation of CS.

A collinearity structure is said to be a collinearity space if:

Let a, b, c, p, q, r be points of it . Then

(i) if $a = b$ or $a = c$ or $b = c$, then $\langle a,b,c\rangle \in$ the collinearity relation of it,

(ii) if $a \neq b$ and $\langle a,b,p\rangle \in$ the collinearity relation of it and $\langle a,b,q\rangle \in$ the collinearity relation of it and $\langle a,b,r\rangle \in$ the collinearity relation of it, then $\langle p,q,r\rangle \in$ the collinearity relation of it.

Next we state the proposition

(6) CS is a collinearity space if and only if for all points a, b, c, p, q, r of CS holds if $a = b$ or $a = c$ or $b = c$, then $\langle a,b,c\rangle \in$ the collinearity relation of CS but if $a \neq b$ and $\langle a,b,p\rangle \in$ the collinearity relation of CS and $\langle a,b,q\rangle \in$ the collinearity relation of CS and $\langle a,b,r\rangle \in$ the collinearity relation of CS, then $\langle p,q,r\rangle \in$ the collinearity relation of CS.

We adopt the following rules: $CLSP$ is a collinearity space and a, b, c, d, p, q, r are points of $CLSP$. We now state several propositions:

(7) If $a = b$ or $a = c$ or $b = c$, then a, b and c are collinear.

(8) If $a \neq b$ and a, b and p are collinear and a, b and q are collinear and a, b and r are collinear, then p, q and r are collinear.

(9) If a, b and c are collinear, then b, a and c are collinear and a, c and b are collinear.

(10) a, b and a are collinear.

(11) If $a \neq b$ and a, b and c are collinear and a, b and d are collinear, then a, c and d are collinear.

(12) If a, b and c are collinear, then b, a and c are collinear.

(13) If a, b and c are collinear, then b, c and a are collinear.

(14) If $p \neq q$ and a, b and p are collinear and a, b and q are collinear and p, q and r are collinear, then a, b and r are collinear.

Let us consider $CLSP$, a, b. The functor $\mathrm{Line}(a,b)$ yields a set and is defined as follows:

$\mathrm{Line}(a,b) = \{p : a,\ b \text{ and } p \text{ are collinear }\}$.

One can prove the following propositions:

(15) $\mathrm{Line}(a,b) = \{p : a,\ b \text{ and } p \text{ are collinear }\}$.

(16) $a \in \mathrm{Line}(a,b)$ and $b \in \mathrm{Line}(a,b)$.

(17) a, b and r are collinear if and only if $r \in \mathrm{Line}(a,b)$.

A collinearity space is said to be a proper collinearity space if:

there exist points a, b, c of it such that a, b and c are not collinear.

The following proposition is true

(18) $CLSP$ is a proper collinearity space if and only if there exist a, b, c such that a, b and c are not collinear.

[2]The propositions (3)–(4) became obvious.

We follow a convention: $CLSP$ will be a proper collinearity space and a, b, p, q, r will be points of $CLSP$. We now state the proposition

(19) For all p, q such that $p \neq q$ there exists r such that p, q and r are not collinear.

Let us consider $CLSP$. A set is called a line of $CLSP$ if:
there exist a, b such that $a \neq b$ and it $= \mathrm{Line}(a, b)$.

The following propositions are true:

(20) For every set X holds X is a line of $CLSP$ if and only if there exist a, b such that $a \neq b$ and $X = \mathrm{Line}(a, b)$.

(21) If $a \neq b$, then $\mathrm{Line}(a, b)$ is a line of $CLSP$.

In the sequel P, Q are lines of $CLSP$. The following propositions are true:

(22) If $a = b$, then $\mathrm{Line}(a, b) =$ the points of $CLSP$.

(23) For every P there exist a, b such that $a \neq b$ and $a \in P$ and $b \in P$.

(24) If $a \neq b$, then there exists P such that $a \in P$ and $b \in P$.

(25) If $p \in P$ and $q \in P$ and $r \in P$, then p, q and r are collinear.

(26) If $P \subseteq Q$, then $P = Q$.

(27) If $p \neq q$ and $p \in P$ and $q \in P$, then $\mathrm{Line}(p, q) \subseteq P$.

(28) If $p \neq q$ and $p \in P$ and $q \in P$, then $\mathrm{Line}(p, q) = P$.

(29) If $p \neq q$ and $p \in P$ and $q \in P$ and $p \in Q$ and $q \in Q$, then $P = Q$.

(30) $P = Q$ or $P \cap Q = \emptyset$ or there exists p such that $P \cap Q = \{p\}$.

(31) If $a \neq b$, then $\mathrm{Line}(a, b) \neq$ the points of $CLSP$.

References

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