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Real Function Continuity [1]

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Summary. The continuity of real functions is discussed. There is a function defined on some domain in real numbers which is continuous in a single point and on a subset of domain of the function. Main properties of real continuous functions are proved. Among them there is the Weierstraß Theorem. Algebraic features for real continuous functions are shown. Lipschitzian functions are introduced. The Lipschitz condition entails continuity.

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The papers [11], [2], [9], [8], [4], [3], [12], [1], [5], [6], [7], and [10] provide the terminology and notation for this paper. For simplicity we adopt the following rules: n is a natural number, X, X_1, Z, Z_1 are sets, s, g, r, p, x_0, x_1, x_2 are real numbers, s_1 is a sequence of real numbers, Y is a subset of $\mathbb{R}$, and f, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$. Let us consider f, x_0. We say that f is continuous in x_0 if and only if:

$x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f(x_0) = \lim(f \cdot s_1)$.

Next we state a number of propositions:

(1) For all f, x_0 holds f is continuous in x_0 if and only if $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f(x_0) = \lim(f \cdot s_1)$.

(2) f is continuous in x_0 if and only if $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ and for every n holds $s_1(n) \neq x_0$ holds $f \cdot s_1$ is convergent and $f(x_0) = \lim(f \cdot s_1)$.

(3) f is continuous in x_0 if and only if $x_0 \in \operatorname{dom} f$ and for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $|x_1 - x_0| < s$ holds $|f(x_1) - f(x_0)| < r$.

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(4) For all f, x_0 holds f is continuous in x_0 if and only if $x_0 \in \operatorname{dom} f$ and for every neighbourhood N_1 of $f(x_0)$ there exists a neighbourhood N of x_0 such that for every x_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f(x_1) \in N_1$.

(5) For all f, x_0 holds f is continuous in x_0 if and only if $x_0 \in \operatorname{dom} f$ and for every neighbourhood N_1 of $f(x_0)$ there exists a neighbourhood N of x_0 such that $f \circ N \subseteq N_1$.

(6) If $x_0 \in \operatorname{dom} f$ and there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$, then f is continuous in x_0.

(7) If f_1 is continuous in x_0 and f_2 is continuous in x_0, then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0 and $f_1 \diamond f_2$ is continuous in x_0.

(8) If f is continuous in x_0, then $r \diamond f$ is continuous in x_0.

(9) If f is continuous in x_0, then $|f|$ is continuous in x_0 and $-f$ is continuous in x_0.

(10) If f is continuous in x_0 and $f(x_0) \neq 0$, then $\frac{1}{f}$ is continuous in x_0.

(11) If f_1 is continuous in x_0 and $f_1(x_0) \neq 0$ and f_2 is continuous in x_0, then $\frac{f_2}{f_1}$ is continuous in x_0.

(12) If f_1 is continuous in x_0 and f_2 is continuous in $f_1(x_0)$, then $f_2 \cdot f_1$ is continuous in x_0.

Let us consider f, X. We say that f is continuous on X if and only if:

$X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

One can prove the following propositions:

(13) For all f, X holds f is continuous on X if and only if $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

(14) For all X, f holds f is continuous on X if and only if $X \subseteq \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 \in X$ holds $f \cdot s_1$ is convergent and $f(\lim s_1) = \lim(f \cdot s_1)$.

(15) f is continuous on X if and only if $X \subseteq \operatorname{dom} f$ and for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $|x_1 - x_0| < s$ holds $|f(x_1) - f(x_0)| < r$.

(16) f is continuous on X if and only if $f \restriction X$ is continuous on X.

(17) If f is continuous on X and $X_1 \subseteq X$, then f is continuous on X_1.

(18) If $x_0 \in \operatorname{dom} f$, then f is continuous on $\{x_0\}$.

(19) For all X, f_1, f_2 such that f_1 is continuous on X and f_2 is continuous on X holds $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X and $f_1 \diamond f_2$ is continuous on X.

(20) For all X, X_1, f_1, f_2 such that f_1 is continuous on X and f_2 is continuous on X_1 holds $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$ and $f_1 \diamond f_2$ is continuous on $X \cap X_1$.

(21) For all r, X, f such that f is continuous on X holds $r \diamond f$ is continuous on X.

(22) If f is continuous on X, then $|f|$ is continuous on X and $-f$ is continuous on X.

(23) If f is continuous on X and $f^{-1}\{0\} = \emptyset$, then $\frac{1}{f}$ is continuous on X.

(24) If f is continuous on X and $(f \restriction X)^{-1}\{0\} = \emptyset$, then $\frac{1}{f}$ is continuous on X.

(25) If f_1 is continuous on X and $f_1{}^{-1}\{0\} = \emptyset$ and f_2 is continuous on X, then $\frac{f_2}{f_1}$ is continuous on X.

(26) If f_1 is continuous on X and f_2 is continuous on $f_1 \circ X$, then $f_2 \cdot f_1$ is continuous on X.

(27) If f_1 is continuous on X and f_2 is continuous on X_1, then $f_2 \cdot f_1$ is continuous on $X \cap f_1{}^{-1} X_1$.

(28) If f is total and for all x_1, x_2 holds $f(x_1 + x_2) = f(x_1) + f(x_2)$ and there exists x_0 such that f is continuous in x_0, then f is continuous on $\mathbb{R}$.

(29) For every f such that $\mathrm{dom}\, f$ is compact and f is continuous on $\mathrm{dom}\, f$ holds $\mathrm{rng}\, f$ is compact.

(30) If $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is continuous on Y, then $f \circ Y$ is compact.

(31) For every f such that $\mathrm{dom}\, f \neq \emptyset$ and $\mathrm{dom}\, f$ is compact and f is continuous on $\mathrm{dom}\, f$ there exist x_1, x_2 such that $x_1 \in \mathrm{dom}\, f$ and $x_2 \in \mathrm{dom}\, f$ and $f(x_1) = \sup(\mathrm{rng}\, f)$ and $f(x_2) = \inf(\mathrm{rng}\, f)$.

(32) For all f, Y such that $Y \neq \emptyset$ and $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is continuous on Y there exist x_1, x_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $f(x_1) = \sup(f \circ Y)$ and $f(x_2) = \inf(f \circ Y)$.

Let us consider f, X. We say that f is Lipschitzian on X if and only if:

$X \subseteq \mathrm{dom}\, f$ and there exists r such that $0 < r$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ holds $|f(x_1) - f(x_2)| \leq r \cdot |x_1 - x_2|$.

One can prove the following propositions:

(33) For every f holds f is Lipschitzian on X if and only if $X \subseteq \mathrm{dom}\, f$ and there exists r such that $0 < r$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ holds $|f(x_1) - f(x_2)| \leq r \cdot |x_1 - x_2|$.

(34) If f is Lipschitzian on X and $X_1 \subseteq X$, then f is Lipschitzian on X_1.

(35) If f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1, then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(36) If f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1, then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

(37) If f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1 and f_1 is bounded on Z and f_2 is bounded on Z_1, then $f_1 \diamond f_2$ is Lipschitzian on $((X \cap Z) \cap X_1) \cap Z_1$.

(38) If f is Lipschitzian on X, then $p \diamond f$ is Lipschitzian on X.

(39) If f is Lipschitzian on X, then $-f$ is Lipschitzian on X and $|f|$ is Lipschitzian on X.

(40) If $X \subseteq \mathrm{dom}\, f$ and f is a constant on X, then f is Lipschitzian on X.

(41) id_Y is Lipschitzian on Y.

(42) If f is Lipschitzian on X, then f is continuous on X.

(43) For every f such that there exists r such that $\mathrm{rng}\, f = \{r\}$ holds f is continuous on $\mathrm{dom}\, f$.

(44) If $X \subseteq \mathrm{dom}\, f$ and f is a constant on X, then f is continuous on X.

(45) For every f such that for every x_0 such that $x_0 \in \mathrm{dom}\, f$ holds $f(x_0) = x_0$ holds f is continuous on $\mathrm{dom}\, f$.

(46) If $f = \mathrm{id}_{\mathrm{dom}\, f}$, then f is continuous on $\mathrm{dom}\, f$.

(47) If $Y \subseteq \mathrm{dom}\, f$ and $f \restriction Y = \mathrm{id}_Y$, then f is continuous on Y.

(48) If $X \subseteq \mathrm{dom}\, f$ and for every x_0 such that $x_0 \in X$ holds $f(x_0) = r \cdot x_0 + p$, then f is continuous on X.

(49) If for every x_0 such that $x_0 \in \mathrm{dom}\, f$ holds $f(x_0) = {x_0}^{\mathbf{2}}$, then f is continuous on $\mathrm{dom}\, f$.

(50) If $X \subseteq \mathrm{dom}\, f$ and for every x_0 such that $x_0 \in X$ holds $f(x_0) = {x_0}^{\mathbf{2}}$, then f is continuous on X.

(51) If for every x_0 such that $x_0 \in \mathrm{dom}\, f$ holds $f(x_0) = |x_0|$, then f is continuous on $\mathrm{dom}\, f$.

(52) If $X \subseteq \mathrm{dom}\, f$ and for every x_0 such that $x_0 \in X$ holds $f(x_0) = |x_0|$, then f is continuous on X.

(53) If $X \subseteq \mathrm{dom}\, f$ and f is monotone on X and there exist p, g such that $p \leq g$ and $f \circ X = [p, g]$, then f is continuous on X.

(54) If $p \leq g$ and $[p, g] \subseteq \mathrm{dom}\, f$ but f is increasing on $[p, g]$ or f is decreasing on $[p, g]$, then $(f \restriction [p, g])^{-1}$ is continuous on $f \circ [p, g]$.

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