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Partial Functions from a Domain to a Domain

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Summary. The value of a partial function from a domain to a domain and a inverse partial function are introduced. The value and inverse function were defined in the article [1], but new definitions are introduced. The basic properties of the value, the inverse partial function, the identity partial function, the composition of partial funtion, the 1−1 partial function, the restriction of a partial function, the image, the inverse image and the graph are proved. Constant partial function are introduced, too.

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The terminology and notation used here are introduced in the following papers: [5], [1], [2], [6], [4], and [3]. For simplicity we follow the rules: x, y are arbitrary, X, Y denote sets, C, D, E denote non-empty sets, SC denotes a subset of C, SD denotes a subset of D, SE denotes a subset of E, c, c_1, c_2 denote elements of C, d denotes an element of D, e denotes an element of E, f, f_1, g denote partial functions from C to D, t denotes a partial function from D to C, s denotes a partial function from D to E, h denotes a partial function from C to E, and F denotes a partial function from D to D. The following proposition is true

(1) x is an element of E if and only if $x \in E$.

Let us consider C, D, f, c. Let us assume that $c \in \operatorname{dom} f$. The functor $f(c)$ yielding an element of D is defined by:

$f(c) = (f\ \textbf{qua}\ \text{a function})(c)$.

Next we state four propositions:

(2) If $c \in \operatorname{dom} f$, then $f(c) = (f\ \textbf{qua}\ \text{a function})(c)$.

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(3) If $\operatorname{dom} f = \operatorname{dom} g$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = g(c)$, then $f = g$.

(4) $y \in \operatorname{rng} f$ if and only if there exists c such that $c \in \operatorname{dom} f$ and $y = f(c)$.

(5) If $c \in \operatorname{dom} f$, then $f(c) \in \operatorname{rng} f$.

Let us consider D, C, f. Then $\operatorname{dom} f$ is a subset of C. Then $\operatorname{rng} f$ is a subset of D.

The following propositions are true:

(6) $h = s \cdot f$ if and only if for every c holds $c \in \operatorname{dom} h$ if and only if $c \in \operatorname{dom} f$ and $f(c) \in \operatorname{dom} s$ and for every c such that $c \in \operatorname{dom} h$ holds $h(c) = s(f(c))$.

(7) $c \in \operatorname{dom}(s \cdot f)$ if and only if $c \in \operatorname{dom} f$ and $f(c) \in \operatorname{dom} s$.

(8) If $c \in \operatorname{dom}(s \cdot f)$, then $(s \cdot f)(c) = s(f(c))$.

(9) If $c \in \operatorname{dom} f$ and $f(c) \in \operatorname{dom} s$, then $(s \cdot f)(c) = s(f(c))$.

(10) If $\operatorname{rng} f \subseteq \operatorname{dom} s$ and $c \in \operatorname{dom} f$, then $(s \cdot f)(c) = s(f(c))$.

(11) If $\operatorname{rng} f = \operatorname{dom} s$ and $c \in \operatorname{dom} f$, then $(s \cdot f)(c) = s(f(c))$.

Let us consider D, SD. Then id_{SD} is a partial function from D to D.

Next we state several propositions:

(12) $F = \operatorname{id}_{SD}$ if and only if $\operatorname{dom} F = SD$ and for every d such that $d \in SD$ holds $F(d) = d$.

(13) If $d \in SD$, then $\operatorname{id}_{SD}(d) = d$.

(14) If $d \in \operatorname{dom} F \cap SD$, then $F(d) = (F \cdot \operatorname{id}_{SD})(d)$.

(15) $d \in \operatorname{dom}(\operatorname{id}_{SD} \cdot F)$ if and only if $d \in \operatorname{dom} F$ and $F(d) \in SD$.

(16) f is one-to-one if and only if for all c_1, c_2 such that $c_1 \in \operatorname{dom} f$ and $c_2 \in \operatorname{dom} f$ and $f(c_1) = f(c_2)$ holds $c_1 = c_2$.

Let us consider C, D, and let f be a partial function from C to D. Let us assume that f is one-to-one. The functor f^{-1} yields a partial function from D to C and is defined as follows:

$f^{-1} = (f\ \mathbf{qua}\ \text{a function})^{-1}$.

One can prove the following propositions:

(17) If f is one-to-one, then for every partial function g from D to C holds $g = f^{-1}$ if and only if $g = (f\ \mathbf{qua}\ \text{a function})^{-1}$.

(18) If f is one-to-one, then for every partial function g from D to C holds $g = f^{-1}$ if and only if $\operatorname{dom} g = \operatorname{rng} f$ and for all d, c holds $d \in \operatorname{rng} f$ and $c = g(d)$ if and only if $c \in \operatorname{dom} f$ and $d = f(c)$.

(19) If f is one-to-one, then $\operatorname{rng} f = \operatorname{dom}(f^{-1})$ and $\operatorname{dom} f = \operatorname{rng}(f^{-1})$.

(20) If f is one-to-one, then $\operatorname{dom}(f^{-1} \cdot f) = \operatorname{dom} f$ and $\operatorname{rng}(f^{-1} \cdot f) = \operatorname{dom} f$.

(21) If f is one-to-one, then $\operatorname{dom}(f \cdot f^{-1}) = \operatorname{rng} f$ and $\operatorname{rng}(f \cdot f^{-1}) = \operatorname{rng} f$.

(22) If f is one-to-one and $c \in \operatorname{dom} f$, then $c = f^{-1}(f(c))$ and $c = (f^{-1} \cdot f)(c)$.

(23) If f is one-to-one and $d \in \operatorname{rng} f$, then $d = f(f^{-1}(d))$ and $d = (f \cdot f^{-1})(d)$.

(24) If f is one-to-one and $\operatorname{dom} f = \operatorname{rng} t$ and $\operatorname{rng} f = \operatorname{dom} t$ and for all c, d such that $c \in \operatorname{dom} f$ and $d \in \operatorname{dom} t$ holds $f(c) = d$ if and only if $t(d) = c$, then $t = f^{-1}$.

(25) If f is one-to-one, then $f^{-1} \cdot f = \mathrm{id}_{\operatorname{dom} f}$ and $f \cdot f^{-1} = \mathrm{id}_{\operatorname{rng} f}$.

(26) If f is one-to-one, then f^{-1} is one-to-one.

(27) If f is one-to-one and $\operatorname{rng} f = \operatorname{dom} s$ and $s \cdot f = \mathrm{id}_{\operatorname{dom} f}$, then $s = f^{-1}$.

(28) If f is one-to-one and $\operatorname{rng} s = \operatorname{dom} f$ and $f \cdot s = \mathrm{id}_{\operatorname{rng} f}$, then $s = f^{-1}$.

(29) If f is one-to-one, then $(f^{-1})^{-1} = f$.

(30) If f is one-to-one and s is one-to-one, then $(s \cdot f)^{-1} = f^{-1} \cdot s^{-1}$.

(31) $(\mathrm{id}_{SC})^{-1} = \mathrm{id}_{SC}$.

Let us consider C, D, f, X. Then $f \restriction X$ is a partial function from C to D.

We now state several propositions:

(32) $g = f \restriction X$ if and only if $\operatorname{dom} g = \operatorname{dom} f \cap X$ and for every c such that $c \in \operatorname{dom} g$ holds $g(c) = f(c)$.

(33) If $c \in \operatorname{dom}(f \restriction X)$, then $(f \restriction X)(c) = f(c)$.

(34) If $c \in \operatorname{dom} f \cap X$, then $(f \restriction X)(c) = f(c)$.

(35) If $c \in \operatorname{dom} f$ and $c \in X$, then $(f \restriction X)(c) = f(c)$.

(36) If $c \in \operatorname{dom} f$ and $c \in X$, then $f(c) \in \operatorname{rng}(f \restriction X)$.

Let us consider C, D, X, f. Then $X \restriction f$ is a partial function from C to D.

The following three propositions are true:

(37) $g = X \restriction f$ if and only if for every c holds $c \in \operatorname{dom} g$ if and only if $c \in \operatorname{dom} f$ and $f(c) \in X$ and for every c such that $c \in \operatorname{dom} g$ holds $g(c) = f(c)$.

(38) $c \in \operatorname{dom}(X \restriction f)$ if and only if $c \in \operatorname{dom} f$ and $f(c) \in X$.

(39) If $c \in \operatorname{dom}(X \restriction f)$, then $(X \restriction f)(c) = f(c)$.

Let us consider C, D, f, X. Then $f \circ X$ is a subset of D.

The following propositions are true:

(40) $SD = f \circ X$ if and only if for every d holds $d \in SD$ if and only if there exists c such that $c \in \operatorname{dom} f$ and $c \in X$ and $d = f(c)$.

(41) $d \in f \circ X$ if and only if there exists c such that $c \in \operatorname{dom} f$ and $c \in X$ and $d = f(c)$.

(42) If $c \in \operatorname{dom} f$, then $f \circ \{c\} = \{f(c)\}$.

(43) If $c_1 \in \operatorname{dom} f$ and $c_2 \in \operatorname{dom} f$, then $f \circ \{c_1, c_2\} = \{f(c_1), f(c_2)\}$.

Let us consider C, D, f, X. Then $f^{-1} X$ is a subset of C.

The following propositions are true:

(44) $SC = f^{-1} X$ if and only if for every c holds $c \in SC$ if and only if $c \in \operatorname{dom} f$ and $f(c) \in X$.

(45) $c \in f^{-1} X$ if and only if $c \in \operatorname{dom} f$ and $f(c) \in X$.

(46) For every f there exists a function g from C into D such that for every c such that $c \in \operatorname{dom} f$ holds $g(c) = f(c)$.

(47) $f \approx g$ if and only if for every c such that $c \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $f(c) = g(c)$.

In this article we present several logical schemes. The scheme *PartFuncExD* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every element d of $\mathcal{A}$ holds $d \in \operatorname{dom} f$ if and only if there exists an element c of $\mathcal{B}$ such that $\mathcal{P}[d, c]$ and for every element d of $\mathcal{A}$ such that $d \in \operatorname{dom} f$ holds $\mathcal{P}[d, f(d)]$

provided the following condition is satisfied:

- for every element d of $\mathcal{A}$ and for all elements c_1, c_2 of $\mathcal{B}$ such that $\mathcal{P}[d, c_1]$ and $\mathcal{P}[d, c_2]$ holds $c_1 = c_2$.

The scheme *LambdaPFD* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every element d of $\mathcal{A}$ holds $d \in \operatorname{dom} f$ if and only if $\mathcal{P}[d]$ and for every element d of $\mathcal{A}$ such that $d \in \operatorname{dom} f$ holds $f(d) = \mathcal{F}(d)$

for all values of the parameters.

The scheme *UnPartFuncD* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a set $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

Let f, g be partial functions from $\mathcal{A}$ to $\mathcal{B}$. Then if $\operatorname{dom} f = \mathcal{C}$ and for every element c of $\mathcal{A}$ such that $c \in \operatorname{dom} f$ holds $f(c) = \mathcal{F}(c)$ and $\operatorname{dom} g = \mathcal{C}$ and for every element c of $\mathcal{A}$ such that $c \in \operatorname{dom} g$ holds $g(c) = \mathcal{F}(c)$, then $f = g$

for all values of the parameters.

Let us consider C, D, SC, d. Then $SC \longmapsto d$ is a partial function from C to D.

The following propositions are true:

(48) If $c \in SC$, then $(SC \longmapsto d)(c) = d$.

(49) If for every c such that $c \in \operatorname{dom} f$ holds $f(c) = d$, then $f = \operatorname{dom} f \longmapsto d$.

(50) If $c \in \operatorname{dom} f$, then $f \cdot (SE \longmapsto c) = SE \longmapsto f(c)$.

(51) id_{SC} is total if and only if $SC = C$.

(52) If $SC \longmapsto d$ is total, then $SC \neq \emptyset$.

(53) $SC \longmapsto d$ is total if and only if $SC = C$.

Let us consider C, D, f, X. We say that f is a constant on X if and only if: there exists d such that for every c such that $c \in X \cap \operatorname{dom} f$ holds $f(c) = d$.

Next we state a number of propositions:

(54) f is a constant on X if and only if there exists d such that for every c such that $c \in X \cap \operatorname{dom} f$ holds $f(c) = d$.

(55) f is a constant on X if and only if for all c_1, c_2 such that $c_1 \in X \cap \operatorname{dom} f$ and $c_2 \in X \cap \operatorname{dom} f$ holds $f(c_1) = f(c_2)$.

(56) If $X \cap \operatorname{dom} f \neq \emptyset$, then f is a constant on X if and only if there exists d such that $\operatorname{rng}(f \restriction X) = \{d\}$.

(57) If f is a constant on X and $Y \subseteq X$, then f is a constant on Y.

(58) If $X \cap \operatorname{dom} f = \emptyset$, then f is a constant on X.

(59) If $f \upharpoonright SC = \operatorname{dom}(f \upharpoonright SC) \longmapsto d$, then f is a constant on SC.

(60) f is a constant on $\{x\}$.

(61) If f is a constant on X and f is a constant on Y and $(X \cap Y) \cap \operatorname{dom} f \neq \emptyset$, then f is a constant on $X \cup Y$.

(62) If f is a constant on Y, then $f \upharpoonright X$ is a constant on Y.

(63) $SC \longmapsto d$ is a constant on SC.

(64) $\operatorname{graph} f \subseteq \operatorname{graph} g$ if and only if $\operatorname{dom} f \subseteq \operatorname{dom} g$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = g(c)$.

(65) $c \in \operatorname{dom} f$ and $d = f(c)$ if and only if $\langle c, d \rangle \in \operatorname{graph} f$.

(66) If $\langle c, e \rangle \in \operatorname{graph}(s \cdot f)$, then $\langle c, f(c) \rangle \in \operatorname{graph} f$ and $\langle f(c), e \rangle \in \operatorname{graph} s$.

(67) If $\operatorname{graph} f = \{\langle c, d \rangle\}$, then $f(c) = d$.

(68) If $\operatorname{dom} f = \{c\}$, then $\operatorname{graph} f = \{\langle c, f(c) \rangle\}$.

(69) If $\operatorname{graph} f_1 = \operatorname{graph} f \cap \operatorname{graph} g$ and $c \in \operatorname{dom} f_1$, then $f_1(c) = f(c)$ and $f_1(c) = g(c)$.

(70) If $c \in \operatorname{dom} f$ and $\operatorname{graph} f_1 = \operatorname{graph} f \cup \operatorname{graph} g$, then $f_1(c) = f(c)$.

(71) If $c \in \operatorname{dom} g$ and $\operatorname{graph} f_1 = \operatorname{graph} f \cup \operatorname{graph} g$, then $f_1(c) = g(c)$.

(72) If $c \in \operatorname{dom} f_1$ and $\operatorname{graph} f_1 = \operatorname{graph} f \cup \operatorname{graph} g$, then $f_1(c) = f(c)$ or $f_1(c) = g(c)$.

(73) $c \in \operatorname{dom} f$ and $c \in SC$ if and only if $\langle c, f(c) \rangle \in \operatorname{graph}(f \upharpoonright SC)$.

(74) $c \in \operatorname{dom} f$ and $f(c) \in SD$ if and only if $\langle c, f(c) \rangle \in \operatorname{graph}(SD \upharpoonright f)$.

(75) $c \in f^{-1}\, SD$ if and only if $\langle c, f(c) \rangle \in \operatorname{graph} f$ and $f(c) \in SD$.

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