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The Lattice of Real Numbers. The Lattice of Real Functions

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Summary. A proof of the fact , that $\langle \mathbb{R}, \max, \min \rangle$ is a lattice (real lattice). Some basic properties (real lattice is distributive and modular) of it are proved. The same is done for the set $\mathbb{R}^A$ with operations: max(f(A)) and min(f(A)), where $\mathbb{R}^A$ means the set of all functions from A (being non-empty set) to $\mathbb{R}$, f is just such a function.

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The articles [4], [1], [3], and [2] provide the terminology and notation for this paper. In the sequel x, y will denote real numbers. Let x be an element of $\mathbb{R}$. The functor $@x$ yielding a real number is defined by:

$@x = x$.

We now state the proposition

(1) For every element x of $\mathbb{R}$ holds $@x = x$.

We now define two new functors. The binary operation $\min_{\mathbb{R}}$ on $\mathbb{R}$ is defined by:

$\min_{\mathbb{R}}(x, y) = \min(x, y)$.

The binary operation $\max_{\mathbb{R}}$ on $\mathbb{R}$ is defined by:

$\max_{\mathbb{R}}(x, y) = \max(x, y)$.

The following propositions are true:

(2) $\min_{\mathbb{R}}(x, y) = \min(x, y)$.

(3) $\max_{\mathbb{R}}(x, y) = \max(x, y)$.

In the sequel p, q will denote elements of the carrier of $\langle \mathbb{R}, \max_{\mathbb{R}}, \min_{\mathbb{R}} \rangle$. Let x be an element of the carrier of $\langle \mathbb{R}, \max_{\mathbb{R}}, \min_{\mathbb{R}} \rangle$. The functor $@x$ yields a real number and is defined by:

$@x = x$.

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Next we state three propositions:

(4) For every element x of the carrier of $\langle \mathbb{R}, \max_{\mathbb{R}}, \min_{\mathbb{R}} \rangle$ holds $@x = x$.

(5) $p \sqcup q = \max_{\mathbb{R}}(p, q)$.

(6) $p \sqcap q = \min_{\mathbb{R}}(p, q)$.

The lattice $\mathbb{R}_{\mathrm{L}}$ is defined as follows:

$\mathbb{R}_{\mathrm{L}} = \langle \mathbb{R}, \max_{\mathbb{R}}, \min_{\mathbb{R}} \rangle$.

One can prove the following proposition

(7) $\mathbb{R}_{\mathrm{L}} = \langle \mathbb{R}, \max_{\mathbb{R}}, \min_{\mathbb{R}} \rangle$.

In the sequel p, q, r denote elements of the carrier of $\mathbb{R}_{\mathrm{L}}$. One can prove the following propositions:

(8) $\max_{\mathbb{R}}(p, q) = \max_{\mathbb{R}}(q, p)$.

(9) $\min_{\mathbb{R}}(p, q) = \min_{\mathbb{R}}(q, p)$.

(10) (i) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(q, r), p)$,

(ii) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(p, q), r)$,

(iii) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(q, p), r)$,

(iv) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(r, p), q)$,

(v) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(r, q), p)$,

(vi) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(p, r), q)$.

(11) (i) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(q, r), p)$,

(ii) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(p, q), r)$,

(iii) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(q, p), r)$,

(iv) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(r, p), q)$,

(v) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(r, q), p)$,

(vi) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(p, r), q)$.

(12) $\max_{\mathbb{R}}(\min_{\mathbb{R}}(p, q), q) = q$ and $\max_{\mathbb{R}}(q, \min_{\mathbb{R}}(p, q)) = q$ and $\max_{\mathbb{R}}(q, \min_{\mathbb{R}}(q, p)) = q$ and $\max_{\mathbb{R}}(\min_{\mathbb{R}}(q, p), q) = q$.

(13) $\min_{\mathbb{R}}(q, \max_{\mathbb{R}}(q, p)) = q$ and $\min_{\mathbb{R}}(\max_{\mathbb{R}}(p, q), q) = q$ and $\min_{\mathbb{R}}(q, \max_{\mathbb{R}}(p, q)) = q$ and $\min_{\mathbb{R}}(\max_{\mathbb{R}}(q, p), q) = q$.

(14) $\min_{\mathbb{R}}(q, \max_{\mathbb{R}}(p, r)) = \max_{\mathbb{R}}(\min_{\mathbb{R}}(q, p), \min_{\mathbb{R}}(q, r))$.

(15) $\mathbb{R}_{\mathrm{L}}$ is a distributive lattice.

In the sequel L will be a distributive lattice. We now state the proposition

(16) L is a modular lattice.

In the sequel A will denote a non-empty set and f, g, h will denote elements of $\mathbb{R}^A$. Let A be a non-empty set, and let x be an element of $\mathbb{R}^A$. The functor $@x$ yielding an element of $\mathbb{R}^A$ **qua** a non-empty set is defined as follows:

$@x = x$.

We now state the proposition

(17) For every element f of $\mathbb{R}^A$ holds $@f = f$.

We now define two new functors. Let us consider A. The functor $\max_{\mathbb{R}^A}$ yielding a binary operation on $\mathbb{R}^A$ is defined by:

$\max_{\mathbb{R}^A}(f, g) = \max_{\mathbb{R}} \circ (f, g)$.

The functor $\min_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined as follows:
$\min_{\mathbb{R}^A}(f, g) = \min_{\mathbb{R}}{}^{\circ}(f, g)$.

Next we state a number of propositions:

(18) $\max_{\mathbb{R}^A}(f, g) = \max_{\mathbb{R}}{}^{\circ}(f, g)$.

(19) $\min_{\mathbb{R}^A}(f, g) = \min_{\mathbb{R}}{}^{\circ}(f, g)$.

(20) $\max_{\mathbb{R}^A}(f, g) = \max_{\mathbb{R}^A}(g, f)$.

(21) $\min_{\mathbb{R}^A}(f, g) = \min_{\mathbb{R}^A}(g, f)$.

(22) $\max_{\mathbb{R}^A}(\max_{\mathbb{R}^A}(f, g), h) = \max_{\mathbb{R}^A}(f, \max_{\mathbb{R}^A}(g, h))$.

(23) $\min_{\mathbb{R}^A}(\min_{\mathbb{R}^A}(f, g), h) = \min_{\mathbb{R}^A}(f, \min_{\mathbb{R}^A}(g, h))$.

(24) $\max_{\mathbb{R}^A}(f, \min_{\mathbb{R}^A}(f, g)) = f$.

(25) $\max_{\mathbb{R}^A}(\min_{\mathbb{R}^A}(f, g), f) = f$.

(26) $\max_{\mathbb{R}^A}(\min_{\mathbb{R}^A}(g, f), f) = f$.

(27) $\max_{\mathbb{R}^A}(f, \min_{\mathbb{R}^A}(g, f)) = f$.

(28) $\min_{\mathbb{R}^A}(f, \max_{\mathbb{R}^A}(f, g)) = f$.

(29) $\min_{\mathbb{R}^A}(f, \max_{\mathbb{R}^A}(g, f)) = f$.

(30) $\min_{\mathbb{R}^A}(\max_{\mathbb{R}^A}(g, f), f) = f$.

(31) $\min_{\mathbb{R}^A}(\max_{\mathbb{R}^A}(f, g), f) = f$.

(32) $\min_{\mathbb{R}^A}(f, \max_{\mathbb{R}^A}(g, h)) = \max_{\mathbb{R}^A}(\min_{\mathbb{R}^A}(f, g), \min_{\mathbb{R}^A}(f, h))$.

We now define two new functors. Let us consider A. The functor $\mathbf{max}_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined by:
$\mathbf{max}_{\mathbb{R}^A}(f, g) = \max_{\mathbb{R}^A}(f, g)$.
The functor $\mathbf{min}_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined as follows:
$\mathbf{min}_{\mathbb{R}^A}(f, g) = \min_{\mathbb{R}^A}(f, g)$.

The following two propositions are true:

(33) $\mathbf{max}_{\mathbb{R}^A}(f, g) = \max_{\mathbb{R}^A}(f, g)$.

(34) $\mathbf{min}_{\mathbb{R}^A}(f, g) = \min_{\mathbb{R}^A}(f, g)$.

In the sequel p, q are elements of the carrier of $\langle \mathbb{R}^A, \mathbf{max}_{\mathbb{R}^A}, \mathbf{min}_{\mathbb{R}^A} \rangle$. Let us consider A, and let x be an element of the carrier of $\langle \mathbb{R}^A, \mathbf{max}_{\mathbb{R}^A}, \mathbf{min}_{\mathbb{R}^A} \rangle$. The functor $@x$ yields an element of $\mathbb{R}^A$ and is defined as follows:
$@x = x$.

The following propositions are true:

(35) $p \sqcup q = \max_{\mathbb{R}^A}(p, q)$.

(36) $p \sqcup q = \mathbf{max}_{\mathbb{R}^A}(p, q)$.

(37) $p \sqcap q = \min_{\mathbb{R}^A}(p, q)$.

(38) $p \sqcap q = \mathbf{min}_{\mathbb{R}^A}(p, q)$.

Let us consider A. The functor $\mathbb{R}^A_{\mathrm{L}}$ yields a lattice and is defined by:
$\mathbb{R}^A_{\mathrm{L}} = \langle \mathbb{R}^A, \mathbf{max}_{\mathbb{R}^A}, \mathbf{min}_{\mathbb{R}^A} \rangle$.

One can prove the following proposition

(39) $\mathbb{R}^A_{\mathrm{L}} = \langle \mathbb{R}^A, \mathbf{max}_{\mathbb{R}^A}, \mathbf{min}_{\mathbb{R}^A} \rangle$.

In the sequel p, q, r will denote elements of the carrier of $\mathbb{R}^A_{\rm L}$. We now state several propositions:

(40) $\mathbf{max}_{\mathbb{R}^A}(p, q) = \mathbf{max}_{\mathbb{R}^A}(q, p)$.

(41) $\mathbf{min}_{\mathbb{R}^A}(p, q) = \mathbf{min}_{\mathbb{R}^A}(q, p)$.

(42) (i) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(q, r), p)$,
(ii) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(p, q), r)$,
(iii) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(q, p), r)$,
(iv) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(r, p), q)$,
(v) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(r, q), p)$,
(vi) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(p, r), q)$.

(43) (i) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(q, r), p)$,
(ii) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(p, q), r)$,
(iii) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(q, p), r)$,
(iv) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(r, p), q)$,
(v) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(r, q), p)$,
(vi) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(p, r), q)$.

(44) $\mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(p, q), q) = q$ and $\mathbf{max}_{\mathbb{R}^A}(q, \mathbf{min}_{\mathbb{R}^A}(p, q)) = q$ and $\mathbf{max}_{\mathbb{R}^A}(q, \mathbf{min}_{\mathbb{R}^A}(q, p)) = q$
and $\mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(q, p), q) = q$.

(45) $\mathbf{min}_{\mathbb{R}^A}(q, \mathbf{max}_{\mathbb{R}^A}(q, p)) = q$ and $\mathbf{min}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(p, q), q) = q$ and $\mathbf{min}_{\mathbb{R}^A}(q, \mathbf{max}_{\mathbb{R}^A}(p, q)) = q$
and $\mathbf{min}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(q, p), q) = q$.

(46) $\mathbf{min}_{\mathbb{R}^A}(q, \mathbf{max}_{\mathbb{R}^A}(p, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(q, p), \mathbf{min}_{\mathbb{R}^A}(q, r))$.

(47) $\mathbb{R}^A_{\rm L}$ is a distributive lattice.

In the sequel F will denote a distributive lattice. We now state the proposition

(48) F is a modular lattice.

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