FORMALIZED MATHEMATICS
Vol.1, No.4, September–October 1990
Université Catholique de Louvain

Introduction to Probability

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Summary. Definitions of Elementary Event and Event in any sample space E are given. Next, the probability of an Event when E is finite is introduced and some properties of this function are investigated. Last part of the paper is devoted to the conditional probability and essential properties of this function (Bayes Theorem).

MML Identifier: RPR_1.

The articles [7], [8], [3], [6], [5], [2], [4], and [1] provide the terminology and notation for this paper. For simplicity we follow the rules: E will denote a non-empty set, a will denote an element of E, A, B, B_1, B_2, B_3, C will denote subsets of E, X, Y will denote sets, and p will denote a finite sequence. Let us consider E. A subset of E is called an elementary event of E if:

it $\subseteq E$ and it $\neq \emptyset$ but $Y \subseteq$ it if and only if $Y = \emptyset$ or $Y =$ it.

In the sequel e, e_1, e_2 will denote elementary events of E. One can prove the following propositions:

(1) If e is an elementary event of E, then $e \subseteq E$.

(2) If e is an elementary event of E, then $e \neq \emptyset$.

(3) For every e such that e is an elementary event of E holds $Y \subseteq e$ if and only if $Y = \emptyset$ or $Y = e$.

(4) e is an elementary event of E if and only if $e \subseteq E$ and $e \neq \emptyset$ but $Y \subseteq e$ if and only if $Y = \emptyset$ or $Y = e$.

(5) If e is an elementary event of E and $e = A \cup B$ and $A \neq B$, then $A = \emptyset$ and $B = e$ or $A = e$ and $B = \emptyset$.

(6) If e is an elementary event of E and $e = A \cup B$, then $A = e$ and $B = e$ or $A = e$ and $B = \emptyset$ or $A = \emptyset$ and $B = e$.

(7) If $a \in E$, then $\{a\}$ is an elementary event of E.

(8) If $\{a\}$ is an elementary event of E, then $a \in E$.

(9) $a \in E$ if and only if $\{a\}$ is an elementary event of E.

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ISSN 0777–4028

(10) If e_1 is an elementary event of E and e_2 is an elementary event of E and $e_1 \subseteq e_2$, then $e_1 = e_2$.

(11) If e is an elementary event of E, then there exists a such that $a \in E$ and $e = \{a\}$.

(12) For every E there exists e such that e is an elementary event of E.

(13) For every E such that e is an elementary event of E holds e is finite.

(14) If e is an elementary event of E, then there exists p such that p is a finite sequence of elements of E and $\operatorname{rng} p = e$ and $\operatorname{len} p = 1$.

Let us consider E. An event of E is a subset of E.

The following propositions are true:

(15) For every subset X of E holds X is an event of E.

(16) $\emptyset$ is an event of E.

(17) E is an event of E.

(18) If A is an event of E and B is an event of E, then $A \cap B$ is an event of E.

(19) If A is an event of E and B is an event of E, then $A \cup B$ is an event of E.

(20) If $A \subseteq B$ and B is an event of E, then A is an event of E.

(21) If A is an event of E, then A^{c} is an event of E.

(22) If e is an elementary event of E and A is an event of E, then $e \cap A = \emptyset$ or $e \cap A = e$.

(23) If A is an event of E and B is an event of E, then $A \setminus B$ is an event of E.

(24) If e is an elementary event of E, then e is an event of E.

(25) If A is an event of E and $A \neq \emptyset$, then there exists e such that e is an elementary event of E and $e \subseteq A$.

(26) If e is an elementary event of E and A is an event of E and $e \subseteq A \cup A^{\mathrm{c}}$, then $e \subseteq A$ or $e \subseteq A^{\mathrm{c}}$.

(27) If e_1 is an elementary event of E and e_2 is an elementary event of E, then $e_1 = e_2$ or $e_1 \cap e_2 = \emptyset$.

Let us consider X, Y. We say that X exclude Y if and only if:
$X \cap Y = \emptyset$.

Next we state several propositions:

(28) X exclude Y if and only if $X \cap Y = \emptyset$.

(29) If X exclude Y, then Y exclude X.

(30) A exclude A^{c}.

(31) For every A holds A exclude $\emptyset$.

(32) A exclude B if and only if $A \setminus B = A$.

(33) $A \cap B$ exclude $A \setminus B$.

(34) $A \cap B$ exclude $A \cap B^{\mathrm{c}}$.

(35) If A exclude B, then A exclude $B \cap C$.

(36) If A exclude B, then $A \cap C$ exclude $B \cap C$.

Let us consider E. Let us assume that E is finite. Let us consider A. The functor $\mathrm{P}(A)$ yields a real number and is defined as follows:

$\mathrm{P}(A) = \frac{\mathrm{card}\, A}{\mathrm{card}\, E}$.

Let us consider E. Then Ω_E is an event of E. Then $\emptyset_E$ is an event of E.

The following propositions are true:

(37) If E is finite and A is an event of E, then $\mathrm{P}(A) = \frac{\mathrm{card}\, A}{\mathrm{card}\, E}$.

(38) If E is finite and e is an elementary event of E, then $\mathrm{P}(e) = \frac{1}{\mathrm{card}\, E}$.

(39) If E is finite, then $\mathrm{P}(\Omega_E) = 1$.

(40) If E is finite, then $\mathrm{P}(\emptyset_E) = 0$.

(41) If E is finite and A is an event of E and B is an event of E and A exclude B, then $\mathrm{P}(A \cap B) = 0$.

(42) If E is finite and A is an event of E, then $\mathrm{P}(A) \leq 1$.

(43) If E is finite and A is an event of E, then $0 \leq \mathrm{P}(A)$.

(44) If E is finite and A is an event of E and B is an event of E and $A \subseteq B$, then $\mathrm{P}(A) \leq \mathrm{P}(B)$.

(46)[1] If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A \cup B) = (\mathrm{P}(A) + \mathrm{P}(B)) - \mathrm{P}(A \cap B)$.

(47) If E is finite and A is an event of E and B is an event of E and A exclude B, then $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.

(48) If E is finite and A is an event of E, then $\mathrm{P}(A) = 1 - \mathrm{P}(A^{\mathrm{c}})$ and $\mathrm{P}(A^{\mathrm{c}}) = 1 - \mathrm{P}(A)$.

(49) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A \setminus B) = \mathrm{P}(A) - \mathrm{P}(A \cap B)$.

(50) If E is finite and A is an event of E and B is an event of E and $B \subseteq A$, then $\mathrm{P}(A \setminus B) = \mathrm{P}(A) - \mathrm{P}(B)$.

(51) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A \cup B) \leq \mathrm{P}(A) + \mathrm{P}(B)$.

(52) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A \setminus B) = \mathrm{P}(A \cap B^{\mathrm{c}})$.

(53) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B^{\mathrm{c}})$.

(54) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A) = \mathrm{P}(A \cup B) - \mathrm{P}(B \setminus A)$.

(55) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A) + \mathrm{P}(A^{\mathrm{c}} \cap B) = \mathrm{P}(B) + \mathrm{P}(B^{\mathrm{c}} \cap A)$.

(56) Suppose E is finite and A is an event of E and B is an event of E and C is an event of E. Then $\mathrm{P}((A \cup B) \cup C) = (((\mathrm{P}(A) + \mathrm{P}(B)) + \mathrm{P}(C)) - ((\mathrm{P}(A \cap B) + \mathrm{P}(A \cap C)) + \mathrm{P}(B \cap C))) + \mathrm{P}((A \cap B) \cap C)$.

[1]The proposition (45) became obvious.

(57) If E is finite and A is an event of E and B is an event of E and C is an event of E and A exclude B and A exclude C and B exclude C, then $\mathrm{P}((A \cup B) \cup C) = (\mathrm{P}(A) + \mathrm{P}(B)) + \mathrm{P}(C)$.

(58) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A) - \mathrm{P}(B) \leq \mathrm{P}(A \setminus B)$.

Let us consider E. Let us assume that E is finite. Let us consider B. Let us assume that $0 < \mathrm{P}(B)$. Let us consider A. The functor $\mathrm{P}(A/B)$ yielding a real number is defined by:

$\mathrm{P}(A/B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.

One can prove the following propositions:

(59) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$, then $\mathrm{P}(A/B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.

(60) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$, then $\mathrm{P}(A \cap B) = \mathrm{P}(A/B) \cdot \mathrm{P}(B)$.

(61) If E is finite and A is an event of E, then $\mathrm{P}(A/\Omega_E) = \mathrm{P}(A)$.

(62) If E is finite, then $\mathrm{P}(\Omega_E/\Omega_E) = 1$.

(63) If E is finite, then $\mathrm{P}(\emptyset_E/\Omega_E) = 0$.

(64) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$, then $\mathrm{P}(A/B) \leq 1$.

(65) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$, then $0 \leq \mathrm{P}(A/B)$.

(66) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$, then $\mathrm{P}(A/B) = 1 - \frac{\mathrm{P}(B \setminus A)}{\mathrm{P}(B)}$.

(67) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$ and $A \subseteq B$, then $\mathrm{P}(A/B) = \frac{\mathrm{P}(A)}{\mathrm{P}(B)}$.

(68) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$ and A exclude B, then $\mathrm{P}(A/B) = 0$.

(69) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(A)$ and $0 < \mathrm{P}(B)$, then $\mathrm{P}(A) \cdot \mathrm{P}(B/A) = \mathrm{P}(B) \cdot \mathrm{P}(A/B)$.

(70) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$, then $\mathrm{P}(A/B) = 1 - \mathrm{P}(A^{\mathrm{c}}/B)$ and $\mathrm{P}(A^{\mathrm{c}}/B) = 1 - \mathrm{P}(A/B)$.

(71) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$ and $B \subseteq A$, then $\mathrm{P}(A/B) = 1$.

(72) If E is finite and B is an event of E and $0 < \mathrm{P}(B)$, then $\mathrm{P}(\Omega_E/B) = 1$.

(73) If E is finite and A is an event of E and $0 < \mathrm{P}(A)$, then $\mathrm{P}(A^{\mathrm{c}}/A) = 0$.

(74) If E is finite and A is an event of E and $\mathrm{P}(A) < 1$, then $\mathrm{P}(A/A^{\mathrm{c}}) = 0$.

(75) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$ and A exclude B, then $\mathrm{P}(A^{\mathrm{c}}/B) = 1$.

(76) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(A)$ and $\mathrm{P}(B) < 1$ and A exclude B, then $\mathrm{P}(A/B^{\mathrm{c}}) = \frac{\mathrm{P}(A)}{1 - \mathrm{P}(B)}$.

(77) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(A)$ and $\mathrm{P}(B) < 1$ and A exclude B, then $\mathrm{P}(A^{\mathrm{c}}/B^{\mathrm{c}}) = 1 - \frac{\mathrm{P}(A)}{1-\mathrm{P}(B)}$.

(78) If E is finite and A is an event of E and B is an event of E and C is an event of E and $0 < \mathrm{P}(B \cap C)$ and $0 < \mathrm{P}(C)$, then $\mathrm{P}((A \cap B) \cap C) = (\mathrm{P}(A/(B \cap C)) \cdot \mathrm{P}(B/C)) \cdot \mathrm{P}(C)$.

(79) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$ and $\mathrm{P}(B) < 1$, then $\mathrm{P}(A) = \mathrm{P}(A/B) \cdot \mathrm{P}(B) + \mathrm{P}(A/B^{\mathrm{c}}) \cdot \mathrm{P}(B^{\mathrm{c}})$.

(80) Suppose E is finite and A is an event of E and B_1 is an event of E and B_2 is an event of E and $0 < \mathrm{P}(B_1)$ and $0 < \mathrm{P}(B_2)$ and $B_1 \cup B_2 = E$ and $B_1 \cap B_2 = \emptyset$. Then $\mathrm{P}(A) = \mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1) + \mathrm{P}(A/B_2) \cdot \mathrm{P}(B_2)$.

(81) Suppose that
(i) E is finite,
(ii) A is an event of E,
(iii) B_1 is an event of E,
(iv) B_2 is an event of E,
(v) B_3 is an event of E,
(vi) $0 < \mathrm{P}(B_1)$,
(vii) $0 < \mathrm{P}(B_2)$,
(viii) $0 < \mathrm{P}(B_3)$,
(ix) $(B_1 \cup B_2) \cup B_3 = E$,
(x) $B_1 \cap B_2 = \emptyset$,
(xi) $B_1 \cap B_3 = \emptyset$,
(xii) $B_2 \cap B_3 = \emptyset$.

Then $\mathrm{P}(A) = (\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1) + \mathrm{P}(A/B_2) \cdot \mathrm{P}(B_2)) + \mathrm{P}(A/B_3) \cdot \mathrm{P}(B_3)$.

(82) Suppose E is finite and A is an event of E and B_1 is an event of E and B_2 is an event of E and $0 < \mathrm{P}(A)$ and $0 < \mathrm{P}(B_1)$ and $0 < \mathrm{P}(B_2)$ and $B_1 \cup B_2 = E$ and $B_1 \cap B_2 = \emptyset$. Then $\mathrm{P}(B_1/A) = \frac{\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1)}{\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1) + \mathrm{P}(A/B_2) \cdot \mathrm{P}(B_2)}$.

(83) Suppose that
(i) E is finite,
(ii) A is an event of E,
(iii) B_1 is an event of E,
(iv) B_2 is an event of E,
(v) B_3 is an event of E,
(vi) $0 < \mathrm{P}(A)$,
(vii) $0 < \mathrm{P}(B_1)$,
(viii) $0 < \mathrm{P}(B_2)$,
(ix) $0 < \mathrm{P}(B_3)$,
(x) $(B_1 \cup B_2) \cup B_3 = E$,
(xi) $B_1 \cap B_2 = \emptyset$,
(xii) $B_1 \cap B_3 = \emptyset$,
(xiii) $B_2 \cap B_3 = \emptyset$.

Then $\mathrm{P}(B_1/A) = \frac{\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1)}{(\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1) + \mathrm{P}(A/B_2) \cdot \mathrm{P}(B_2)) + \mathrm{P}(A/B_3) \cdot \mathrm{P}(B_3)}$.

Let us consider E, A, B. We say that A and B are independent if and only if:

$\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$.

The following propositions are true:

(84) A and B are independent if and only if $\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$.

(85) If A and B are independent, then B and A are independent.

(86) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$ and A and B are independent, then $\mathrm{P}(A/B) = \mathrm{P}(A)$.

(87) If E is finite and A is an event of E and B is an event of E and $\mathrm{P}(B) = 0$, then A and B are independent.

(88) If E is finite and A is an event of E and B is an event of E and A and B are independent, then A^{c} and B are independent.

(89) If E is finite and A is an event of E and B is an event of E and A exclude B and A and B are independent, then $\mathrm{P}(A) = 0$ or $\mathrm{P}(B) = 0$.

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Received June 13, 1990