FORMALIZED MATHEMATICS
Vol.1, No.4, September–October 1990
Université Catholique de Louvain

The Sum and Product of Finite Sequences of Real Numbers

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Summary. Some operations on the set of n-tuples of real numbers are introduced. Addition, difference of such n-tuples, complement of a n-tuple and multiplication of these by real numbers are defined. In these definitions more general properties of binary operations applied to finite sequences from [3] are used. Then the fact that certain properties are satisfied by those operations is demonstrated directly from [3]. Moreover some properties can be recognized as being those of real vector space. Multiplication of n-tuples of real numbers and square power of n-tuple of real numbers using for notation of some properties of finite sums and products of real numbers are defined, followed by definitions of the finite sum and product of n-tuples of real numbers using notions and properties introduced in [7]. A number of propositions and theorems on sum and product of finite sequences of real numbers are proved. As a additional properties there are proved some properties of real numbers and set representations of binary operations on real numbers.

MML Identifier: `RVSUM_1`.

The papers [8], [12], [5], [6], [1], [2], [13], [10], [9], [11], [4], [3], and [7] provide the terminology and notation for this paper. For simplicity we follow the rules: i, j, k are natural numbers, r, r', r_1, r_2, r_3 are real numbers, x is an element of $\mathbb{R}$, F, F_1, F_2 are finite sequences of elements of $\mathbb{R}$, and R, R_1, R_2, R_3 are elements of $\mathbb{R}^i$. Next we state the proposition

(1) $-(r_1 + r_2) = (-r_1) + (-r_2)$.

Let us consider x. The functor $@x$ yields a real number and is defined by: $@x = x$.

The following propositions are true:

(2) $@x = x$.

[1]Supported by RPBP.III-24.C1.

ISSN 0777–4028

(3) 0 is a unity w.r.t. $+_{\mathbb{R}}$.

(4) $\mathbf{1}_{+_{\mathbb{R}}} = 0$.

(5) $+_{\mathbb{R}}$ has a unity.

(6) $+_{\mathbb{R}}$ is commutative.

(7) $+_{\mathbb{R}}$ is associative.

The binary operation $-_{\mathbb{R}}$ on $\mathbb{R}$ is defined as follows:
$-_{\mathbb{R}} = +_{\mathbb{R}} \circ (\mathrm{id}_{\mathbb{R}}, -_{\mathbb{R}})$.

We now state two propositions:

(8) $-_{\mathbb{R}} = +_{\mathbb{R}} \circ (\mathrm{id}_{\mathbb{R}}, -_{\mathbb{R}})$.

(9) $-_{\mathbb{R}}(r_1, r_2) = r_1 - r_2$.

The unary operation $\mathrm{sqr}_{\mathbb{R}}$ on $\mathbb{R}$ is defined as follows:
for every r holds $\mathrm{sqr}_{\mathbb{R}}(r) = r^{\mathbf{2}}$.

The following propositions are true:

(10) For every unary operation u on $\mathbb{R}$ holds $u = \mathrm{sqr}_{\mathbb{R}}$ if and only if for every r holds $u(r) = r^{\mathbf{2}}$.

(11) $\cdot_{\mathbb{R}}$ is commutative.

(12) $\cdot_{\mathbb{R}}$ is associative.

(13) 1 is a unity w.r.t. $\cdot_{\mathbb{R}}$.

(14) $\mathbf{1}_{\cdot_{\mathbb{R}}} = 1$.

(15) $\cdot_{\mathbb{R}}$ has a unity.

(16) $\cdot_{\mathbb{R}}$ is distributive w.r.t. $+_{\mathbb{R}}$.

(17) $\mathrm{sqr}_{\mathbb{R}}$ is distributive w.r.t. $\cdot_{\mathbb{R}}$.

Let us consider x. The functor $\cdot_{\mathbb{R}}^{x}$ yielding a unary operation on $\mathbb{R}$ is defined by:
$\cdot_{\mathbb{R}}^{x} = \cdot_{\mathbb{R}}{}^{\circ}(x, \mathrm{id}_{\mathbb{R}})$.

Next we state several propositions:

(18) $\cdot_{\mathbb{R}}^{x} = \cdot_{\mathbb{R}}{}^{\circ}(x, \mathrm{id}_{\mathbb{R}})$.

(19) $\cdot_{\mathbb{R}}^{r}(x) = r \cdot x$.

(20) $\cdot_{\mathbb{R}}^{r}$ is distributive w.r.t. $+_{\mathbb{R}}$.

(21) $-_{\mathbb{R}}$ is an inverse operation w.r.t. $+_{\mathbb{R}}$.

(22) $+_{\mathbb{R}}$ has an inverse operation.

(23) The inverse operation w.r.t.$+_{\mathbb{R}} = -_{\mathbb{R}}$.

(24) $-_{\mathbb{R}}$ is distributive w.r.t. $+_{\mathbb{R}}$.

Let us consider F_1, F_2. The functor $F_1 + F_2$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:
$F_1 + F_2 = +_{\mathbb{R}}{}^{\circ}(F_1, F_2)$.

We now state two propositions:

(25) $F_1 + F_2 = +_{\mathbb{R}}{}^{\circ}(F_1, F_2)$.

(26) If $i \in \mathrm{Seg}(\mathrm{len}(F_1 + F_2))$ and $r_1 = F_1(i)$ and $r_2 = F_2(i)$, then $(F_1 + F_2)(i) = r_1 + r_2$.

Let us consider i, R_1, R_2. Then $R_1 + R_2$ is an element of $\mathbb{R}^i$.

We now state several propositions:

(27) If $j \in \operatorname{Seg} i$ and $r_1 = R_1(j)$ and $r_2 = R_2(j)$, then $(R_1+R_2)(j) = r_1+r_2$.

(28) $\varepsilon_{\mathbb{R}} + F = \varepsilon_{\mathbb{R}}$ and $F + \varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(29) $\langle r_1 \rangle + \langle r_2 \rangle = \langle r_1 + r_2 \rangle$.

(30) $(i \longmapsto r_1) + (i \longmapsto r_2) = i \longmapsto r_1 + r_2$.

(31) $R_1 + R_2 = R_2 + R_1$.

(32) $R_1 + (R_2 + R_3) = (R_1 + R_2) + R_3$.

(33) $R + (i \longmapsto (0\,\textbf{qua}\text{ a real number})) = R$ and $R = (i \longmapsto (0\,\textbf{qua}\text{ a real number})) + R$.

Let us consider F. The functor $-F$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

$-F = -_{\mathbb{R}} \cdot F$.

We now state two propositions:

(34) $-F = -_{\mathbb{R}} \cdot F$.

(35) If $i \in \operatorname{Seg}(\operatorname{len}(-F))$ and $r = F(i)$, then $(-F)(i) = -r$.

Let us consider i, R. Then $-R$ is an element of $\mathbb{R}^i$.

The following propositions are true:

(36) If $j \in \operatorname{Seg} i$ and $r = R(j)$, then $(-R)(j) = -r$.

(37) $-\varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(38) $-\langle r \rangle = \langle -r \rangle$.

(39) $-(i \longmapsto r) = i \longmapsto -r$.

(40) $R + (-R) = i \longmapsto 0$ and $(-R) + R = i \longmapsto 0$.

(41) If $R_1 + R_2 = i \longmapsto 0$, then $R_1 = -R_2$ and $R_2 = -R_1$.

(42) $-(-R) = R$.

(43) If $-R_1 = -R_2$, then $R_1 = R_2$.

(44) If $R_1 + R = R_2 + R$ or $R_1 + R = R + R_2$, then $R_1 = R_2$.

(45) $-(R_1 + R_2) = (-R_1) + (-R_2)$.

Let us consider F_1, F_2. The functor $F_1 - F_2$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

$F_1 - F_2 = -_{\mathbb{R}}{}^{\circ}(F_1, F_2)$.

The following two propositions are true:

(46) $F_1 - F_2 = -_{\mathbb{R}}{}^{\circ}(F_1, F_2)$.

(47) If $i \in \operatorname{Seg}(\operatorname{len}(F_1 - F_2))$ and $r_1 = F_1(i)$ and $r_2 = F_2(i)$, then $(F_1 - F_2)(i) = r_1 - r_2$.

Let us consider i, R_1, R_2. Then $R_1 - R_2$ is an element of $\mathbb{R}^i$.

One can prove the following propositions:

(48) If $j \in \operatorname{Seg} i$ and $r_1 = R_1(j)$ and $r_2 = R_2(j)$, then $(R_1-R_2)(j) = r_1-r_2$.

(49) $\varepsilon_{\mathbb{R}} - F = \varepsilon_{\mathbb{R}}$ and $F - \varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(50) $\langle r_1 \rangle - \langle r_2 \rangle = \langle r_1 - r_2 \rangle$.

(51) $(i \longmapsto r_1) - (i \longmapsto r_2) = i \longmapsto r_1 - r_2$.

(52) $R_1 - R_2 = R_1 + (-R_2)$.

(53) $R - (i \longmapsto (0\,\mathbf{qua}\ \text{a real number})) = R$.

(54) $(i \longmapsto (0\,\mathbf{qua}\ \text{a real number})) - R = -R$.

(55) $R_1 - (-R_2) = R_1 + R_2$.

(56) $-(R_1 - R_2) = R_2 - R_1$.

(57) $-(R_1 - R_2) = (-R_1) + R_2$.

(58) $R - R = i \longmapsto 0$.

(59) If $R_1 - R_2 = i \longmapsto 0$, then $R_1 = R_2$.

(60) $(R_1 - R_2) - R_3 = R_1 - (R_2 + R_3)$.

(61) $R_1 + (R_2 - R_3) = (R_1 + R_2) - R_3$.

(62) $R_1 - (R_2 - R_3) = (R_1 - R_2) + R_3$.

(63) $R_1 = (R_1 + R) - R$.

(64) $R_1 = (R_1 - R) + R$.

Let us consider r, F. The functor $r \cdot F$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

$r \cdot F = \cdot^{r}_{\mathbb{R}} \cdot F$.

We now state two propositions:

(65) $r \cdot F = \cdot^{r}_{\mathbb{R}} \cdot F$.

(66) If $i \in \operatorname{Seg}(\operatorname{len}(r \cdot F))$ and $r' = F(i)$, then $(r \cdot F)(i) = r \cdot r'$.

Let us consider i, r, R. Then $r \cdot R$ is an element of $\mathbb{R}^i$.

Next we state a number of propositions:

(67) If $j \in \operatorname{Seg} i$ and $r' = R(j)$, then $(r \cdot R)(j) = r \cdot r'$.

(68) $r \cdot \varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(69) $r \cdot \langle r_1 \rangle = \langle r \cdot r_1 \rangle$.

(70) $r_1 \cdot (i \longmapsto r_2) = i \longmapsto r_1 \cdot r_2$.

(71) $(r_1 \cdot r_2) \cdot R = r_1 \cdot (r_2 \cdot R)$.

(72) $(r_1 + r_2) \cdot R = r_1 \cdot R + r_2 \cdot R$.

(73) $r \cdot (R_1 + R_2) = r \cdot R_1 + r \cdot R_2$.

(74) $1 \cdot R = R$.

(75) $0 \cdot R = i \longmapsto 0$.

(76) $(-1) \cdot R = -R$.

Let us consider F. The functor ${}^2F$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

${}^2F = \operatorname{sqr}_{\mathbb{R}} \cdot F$.

Next we state two propositions:

(77) ${}^2F = \operatorname{sqr}_{\mathbb{R}} \cdot F$.

(78) If $i \in \operatorname{Seg}(\operatorname{len}({}^2F))$ and $r = F(i)$, then ${}^2F(i) = r^{\mathbf{2}}$.

Let us consider i, R. Then ${}^2R$ is an element of $\mathbb{R}^i$.

Next we state several propositions:

(79) If $j \in \operatorname{Seg} i$ and $r = R(j)$, then ${}^2R(j) = r^{\mathbf{2}}$.

(80) ${}^2\varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(81) ${}^2\langle r \rangle = \langle r^{\mathbf{2}} \rangle$.

(82) ${}^2(i \longmapsto r) = i \longmapsto r^{\mathbf{2}}$.

(83) ${}^2(-R) = {}^2R$.

(84) ${}^2(r \cdot R) = r^{\mathbf{2}} \cdot {}^2R$.

Let us consider F_1, F_2. The functor $F_1 \bullet F_2$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

$F_1 \bullet F_2 = \cdot_{\mathbb{R}}{}^{\circ}(F_1, F_2)$.

One can prove the following two propositions:

(85) $F_1 \bullet F_2 = \cdot_{\mathbb{R}}{}^{\circ}(F_1, F_2)$.

(86) If $i \in \operatorname{Seg}(\operatorname{len}(F_1 \bullet F_2))$ and $r_1 = F_1(i)$ and $r_2 = F_2(i)$, then $F_1 \bullet F_2(i) = r_1 \cdot r_2$.

Let us consider i, R_1, R_2. Then $R_1 \bullet R_2$ is an element of $\mathbb{R}^i$.

The following propositions are true:

(87) If $j \in \operatorname{Seg} i$ and $r_1 = R_1(j)$ and $r_2 = R_2(j)$, then $R_1 \bullet R_2(j) = r_1 \cdot r_2$.

(88) $\varepsilon_{\mathbb{R}} \bullet F = \varepsilon_{\mathbb{R}}$ and $F \bullet \varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(89) $\langle r_1 \rangle \bullet \langle r_2 \rangle = \langle r_1 \cdot r_2 \rangle$.

(90) $R_1 \bullet R_2 = R_2 \bullet R_1$.

(91) $R_1 \bullet (R_2 \bullet R_3) = (R_1 \bullet R_2) \bullet R_3$.

(92) $(i \longmapsto r) \bullet R = r \cdot R$ and $R \bullet (i \longmapsto r) = r \cdot R$.

(93) $(i \longmapsto r_1) \bullet (i \longmapsto r_2) = i \longmapsto r_1 \cdot r_2$.

(94) $r \cdot R_1 \bullet R_2 = (r \cdot R_1) \bullet R_2$.

(95) $r \cdot R_1 \bullet R_2 = (r \cdot R_1) \bullet R_2$ and $r \cdot R_1 \bullet R_2 = R_1 \bullet (r \cdot R_2)$.

(96) $r \cdot R = (i \longmapsto r) \bullet R$.

(97) ${}^2R = R \bullet R$.

(98) ${}^2(R_1 + R_2) = ({}^2R_1 + 2 \cdot R_1 \bullet R_2) + {}^2R_2$.

(99) ${}^2(R_1 - R_2) = ({}^2R_1 - 2 \cdot R_1 \bullet R_2) + {}^2R_2$.

(100) ${}^2(R_1 \bullet R_2) = ({}^2R_1) \bullet ({}^2R_2)$.

Let F be a finite sequence of elements of $\mathbb{R}$. The functor $\sum F$ yields a real number and is defined by:

$\sum F = +_{\mathbb{R}} \circledast F$.

One can prove the following propositions:

(101) $\sum F = +_{\mathbb{R}} \circledast F$.

(102) $\sum \varepsilon_{\mathbb{R}} = 0$.

(103) $\sum \langle r \rangle = r$.

(104) $\sum (F \frown \langle r \rangle) = \sum F + r$.

(105) $\sum(F_1 \frown F_2) = \sum F_1 + \sum F_2$.

(106) $\sum(\langle r\rangle \frown F) = r + \sum F$.

(107) $\sum\langle r_1, r_2\rangle = r_1 + r_2$.

(108) $\sum\langle r_1, r_2, r_3\rangle = (r_1 + r_2) + r_3$.

(109) For every element R of $\mathbb{R}^0$ holds $\sum R = 0$.

(110) $\sum(i \longmapsto r) = i \cdot r$.

(111) $\sum(i \longmapsto (0\,\mathbf{qua}\,\text{a real number})) = 0$.

(112) If for all j, r_1, r_2 such that $j \in \operatorname{Seg} i$ and $r_1 = R_1(j)$ and $r_2 = R_2(j)$ holds $r_1 \leq r_2$, then $\sum R_1 \leq \sum R_2$.

(113) Suppose for all j, r_1, r_2 such that $j \in \operatorname{Seg} i$ and $r_1 = R_1(j)$ and $r_2 = R_2(j)$ holds $r_1 \leq r_2$ and there exist j, r_1, r_2 such that $j \in \operatorname{Seg} i$ and $r_1 = R_1(j)$ and $r_2 = R_2(j)$ and $r_1 < r_2$. Then $\sum R_1 < \sum R_2$.

(114) If for all i, r such that $i \in \operatorname{Seg}(\operatorname{len} F)$ and $r = F(i)$ holds $0 \leq r$, then $0 \leq \sum F$.

(115) If for all i, r such that $i \in \operatorname{Seg}(\operatorname{len} F)$ and $r = F(i)$ holds $0 \leq r$ and there exist i, r such that $i \in \operatorname{Seg}(\operatorname{len} F)$ and $r = F(i)$ and $0 < r$, then $0 < \sum F$.

(116) $0 \leq \sum(^2F)$.

(117) $\sum(r \cdot F) = r \cdot \sum F$.

(118) $\sum(-F) = -\sum F$.

(119) $\sum(R_1 + R_2) = \sum R_1 + \sum R_2$.

(120) $\sum(R_1 - R_2) = \sum R_1 - \sum R_2$.

(121) If $\sum(^2R) = 0$, then $R = i \longmapsto 0$.

(122) $(\sum(R_1 \bullet R_2))^{\mathbf{2}} \leq \sum(^2R_1) \cdot \sum(^2R_2)$.

Let F be a finite sequence of elements of $\mathbb{R}$. The functor $\prod F$ yields a real number and is defined as follows:

$\prod F = \cdot_{\mathbb{R}} \circledast F$.

Next we state a number of propositions:

(123) $\prod F = \cdot_{\mathbb{R}} \circledast F$.

(124) $\prod \varepsilon_{\mathbb{R}} = 1$.

(125) $\prod\langle r\rangle = r$.

(126) $\prod(F \frown \langle r\rangle) = \prod F \cdot r$.

(127) $\prod(F_1 \frown F_2) = \prod F_1 \cdot \prod F_2$.

(128) $\prod(\langle r\rangle \frown F) = r \cdot \prod F$.

(129) $\prod\langle r_1, r_2\rangle = r_1 \cdot r_2$.

(130) $\prod\langle r_1, r_2, r_3\rangle = (r_1 \cdot r_2) \cdot r_3$.

(131) For every element R of $\mathbb{R}^0$ holds $\prod R = 1$.

(132) $\prod(i \longmapsto (1\,\mathbf{qua}\,\text{a real number})) = 1$.

(133) There exists k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ and $F(k) = 0$ if and only if $\prod F = 0$.

(134) $\prod(i + j \longmapsto r) = \prod(i \longmapsto r) \cdot \prod(j \longmapsto r)$.

(135) $\prod(i \cdot j \longmapsto r) = \prod(j \longmapsto \prod(i \longmapsto r))$.

(136) $\prod(i \longmapsto r_1 \cdot r_2) = \prod(i \longmapsto r_1) \cdot \prod(i \longmapsto r_2)$.

(137) $\prod(R_1 \bullet R_2) = \prod R_1 \cdot \prod R_2$.

(138) $\prod(r \cdot R) = \prod(i \longmapsto r) \cdot \prod R$.

(139) $\prod({}^2R) = (\prod R)^{\mathbf{2}}$.

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Received May 11, 1990