FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Projective Spaces - part V

Wojciech Leończuk[1]
Warsaw University
Białystok

Krzysztof Prażmowski[2]
Warsaw University
Białystok

Summary. In the classes of projective spaces, defined in terms of collinearity, introduced in the article [3], we distinguish the subclasses of Pappian projective structures. As examples of these objects we consider analytical projective spaces defined over suitable real linear spaces; analytical counterpart of the Pappus Axiom is proved without any assumption on the dimension of the underlying linear space.

MML Identifier: ANPROJ_6.

The terminology and notation used in this paper are introduced in the following papers: [1], [5], [2], [3], and [4]. We follow a convention: V will denote a real linear space, o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 will denote vectors of V, and a, b, c will denote real numbers. Let us consider V, o, p_1, p_2, p_3, q_1, q_2, q_3. We say that o, p_1, p_2, p_3, q_1, q_2, and q_3 lie on an angle if and only if:

(Def.1) o, p_1 and q_1 are not lineary dependent and o, p_1 and p_2 are lineary dependent and o, p_1 and p_3 are lineary dependent and o, q_1 and q_2 are lineary dependent and o, q_1 and q_3 are lineary dependent.

One can prove the following proposition

(1) o, p_1, p_2, p_3, q_1, q_2, and q_3 lie on an angle if and only if o, p_1 and q_1 are not lineary dependent and o, p_1 and p_2 are lineary dependent and o, p_1 and p_3 are lineary dependent and o, q_1 and q_2 are lineary dependent and o, q_1 and q_3 are lineary dependent.

Let us consider V, o, p_1, p_2, p_3, q_1, q_2, q_3. We say that o, p_1, p_2, p_3, q_1, q_2, q_3 are half-mutually not proportional if and only if:

(Def.2) o and p_2 are not proportional and o and p_3 are not proportional and o and q_2 are not proportional and o and q_3 are not proportional and p_1 and p_2 are not proportional and p_1 and p_3 are not proportional and q_1 and

[1]Supported by RPBP.III-24.C6.
[2]Supported by RPBP.III-24.C2.

ISSN 0777–4028

q_2 are not proportional and q_1 and q_3 are not proportional and p_2 and p_3 are not proportional and q_2 and q_3 are not proportional.

Next we state two propositions:

(2) o, p_1, p_2, p_3, q_1, q_2, q_3 are half-mutually not proportional if and only if the following conditions are satisfied:

(i) o and p_2 are not proportional,
(ii) o and p_3 are not proportional,
(iii) o and q_2 are not proportional,
(iv) o and q_3 are not proportional,
(v) p_1 and p_2 are not proportional,
(vi) p_1 and p_3 are not proportional,
(vii) q_1 and q_2 are not proportional,
(viii) q_1 and q_3 are not proportional,
(ix) p_2 and p_3 are not proportional,
(x) q_2 and q_3 are not proportional.

(3) Suppose that

(i) o is a proper vector,
(ii) p_1, p_2 and p_3 are proper vectors,
(iii) q_1, q_2 and q_3 are proper vectors,
(iv) r_1, r_2 and r_3 are proper vectors,
(v) o, p_1, p_2, p_3, q_1, q_2, and q_3 lie on an angle,
(vi) o, p_1, p_2, p_3, q_1, q_2, q_3 are half-mutually not proportional,
(vii) p_1, q_2 and r_3 are lineary dependent,
(viii) q_1, p_2 and r_3 are lineary dependent,
(ix) p_1, q_3 and r_2 are lineary dependent,
(x) p_3, q_1 and r_2 are lineary dependent,
(xi) p_2, q_3 and r_1 are lineary dependent,
(xii) p_3, q_2 and r_1 are lineary dependent.

Then r_1, r_2 and r_3 are lineary dependent.

We adopt the following convention: V will denote a non-trivial real linear space and o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 will denote elements of the points of the projective space over V. The following proposition is true

(4) Suppose that

(i) $o \neq p_2$,
(ii) $o \neq p_3$,
(iii) $p_2 \neq p_3$,
(iv) $p_1 \neq p_2$,
(v) $p_1 \neq p_3$,
(vi) $o \neq q_2$,
(vii) $o \neq q_3$,
(viii) $q_2 \neq q_3$,
(ix) $q_1 \neq q_2$,
(x) $q_1 \neq q_3$,
(xi) o, p_1 and q_1 are not collinear,

(xii) o, p_1 and p_2 are collinear,
(xiii) o, p_1 and p_3 are collinear,
(xiv) o, q_1 and q_2 are collinear,
(xv) o, q_1 and q_3 are collinear,
(xvi) p_1, q_2 and r_3 are collinear,
(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.

Then r_1, r_2 and r_3 are collinear.

In the sequel u, v, w, y are vectors of V. A projective space defined in terms of collinearity is said to be a Pappian projective space defined in terms of collinearity if:

(Def.3) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq p_2$,
(ii) $o \neq p_3$,
(iii) $p_2 \neq p_3$,
(iv) $p_1 \neq p_2$,
(v) $p_1 \neq p_3$,
(vi) $o \neq q_2$,
(vii) $o \neq q_3$,
(viii) $q_2 \neq q_3$,
(ix) $q_1 \neq q_2$,
(x) $q_1 \neq q_3$,
(xi) o, p_1 and q_1 are not collinear,
(xii) o, p_1 and p_2 are collinear,
(xiii) o, p_1 and p_3 are collinear,
(xiv) o, q_1 and q_2 are collinear,
(xv) o, q_1 and q_3 are collinear,
(xvi) p_1, q_2 and r_3 are collinear,
(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state three propositions:

(5) Let C_1 be a projective space defined in terms of collinearity. Then C_1 is a Pappian projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not

collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(6) If there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then the projective space over V is a Pappian projective space defined in terms of collinearity.

(7) Let C_1 be a collinearity structure. Then C_1 is a Pappian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(vi) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

A Fanoian projective space defined in terms of collinearity is said to be a Fano-Pappian projective space defined in terms of collinearity if:

(Def.4) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq p_2$,

(ii) $o \neq p_3$,

(iii) $p_2 \neq p_3$,

(iv) $p_1 \neq p_2$,

(v) $p_1 \neq p_3$,

(vi) $o \neq q_2$,

(vii) $o \neq q_3$,

(viii) $q_2 \neq q_3$,

(ix) $q_1 \neq q_2$,
(x) $q_1 \neq q_3$,
(xi) o, p_1 and q_1 are not collinear,
(xii) o, p_1 and p_2 are collinear,
(xiii) o, p_1 and p_3 are collinear,
(xiv) o, q_1 and q_2 are collinear,
(xv) o, q_1 and q_3 are collinear,
(xvi) p_1, q_2 and r_3 are collinear,
(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state four propositions:

(8) Let C_1 be a Fanoian projective space defined in terms of collinearity. Then C_1 is a Fano-Pappian projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(9) If there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then the projective space over V is a Fano-Pappian projective space defined in terms of collinearity.

(10) Let C_1 be a collinearity structure. Then C_1 is a Fano-Pappian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(vi) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(11) Let C_1 be a collinearity structure. Then C_1 is a Fano-Pappian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) C_1 is a Pappian projective space defined in terms of collinearity,

(ii) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

A projective plane defined in terms of collinearity is called a Pappian projective plane defined in terms of collinearity if:

(Def.5) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq p_2$,

(ii) $o \neq p_3$,

(iii) $p_2 \neq p_3$,

(iv) $p_1 \neq p_2$,

(v) $p_1 \neq p_3$,

(vi) $o \neq q_2$,

(vii) $o \neq q_3$,

(viii) $q_2 \neq q_3$,

(ix) $q_1 \neq q_2$,

(x) $q_1 \neq q_3$,

(xi) o, p_1 and q_1 are not collinear,

(xii) o, p_1 and p_2 are collinear,

(xiii) o, p_1 and p_3 are collinear,

(xiv) o, q_1 and q_2 are collinear,

(xv) o, q_1 and q_3 are collinear,

(xvi) p_1, q_2 and r_3 are collinear,

(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.
Then r_1, r_2 and r_3 are collinear.

We now state four propositions:

(12) Let C_1 be a projective plane defined in terms of collinearity. Then C_1 is a Pappian projective plane defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(13) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then the projective space over V is a Pappian projective plane defined in terms of collinearity.

(14) Let C_1 be a collinearity structure. Then C_1 is a Pappian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(v) for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2

are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(15) For every C_1 being a collinearity structure holds C_1 is a Pappian projective plane defined in terms of collinearity if and only if C_1 is a Pappian projective space defined in terms of collinearity and for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear.

A Fanoian projective plane defined in terms of collinearity is called a Fano-Pappian projective plane defined in terms of collinearity if:

(Def.6) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq p_2$,
(ii) $o \neq p_3$,
(iii) $p_2 \neq p_3$,
(iv) $p_1 \neq p_2$,
(v) $p_1 \neq p_3$,
(vi) $o \neq q_2$,
(vii) $o \neq q_3$,
(viii) $q_2 \neq q_3$,
(ix) $q_1 \neq q_2$,
(x) $q_1 \neq q_3$,
(xi) o, p_1 and q_1 are not collinear,
(xii) o, p_1 and p_2 are collinear,
(xiii) o, p_1 and p_3 are collinear,
(xiv) o, q_1 and q_2 are collinear,
(xv) o, q_1 and q_3 are collinear,
(xvi) p_1, q_2 and r_3 are collinear,
(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state several propositions:

(16) Let C_1 be a Fanoian projective plane defined in terms of collinearity. Then C_1 is a Fano-Pappian projective plane defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(17) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then the projective space over V is a Fano-Pappian projective plane defined in terms of collinearity.

(18) Let C_1 be a collinearity structure. Then C_1 is a Fano-Pappian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(v) for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(19) Let C_1 be a collinearity structure. Then C_1 is a Fano-Pappian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) C_1 is a Pappian projective plane defined in terms of collinearity,

(ii) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or

r_2, r_1 and q_1 are collinear.

(20) For every C_1 being a collinearity structure holds C_1 is a Fano-Pappian projective plane defined in terms of collinearity if and only if C_1 is a Fano-Pappian projective space defined in terms of collinearity and for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Wojciech Leończuk and Krzysztof Prażmowski. A construction of analytical projective space. *Formalized Mathematics*, 1(**4**):761–766, 1990.

[3] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces - part I. *Formalized Mathematics*, 1(**4**):767–776, 1990.

[4] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces - part III. *Formalized Mathematics*, 1(**5**):909–918, 1990.

[5] Wojciech Skaba. The collinearity structure. *Formalized Mathematics*, 1(**4**):657–659, 1990.

Received August 10, 1990