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Factorial and Newton coeffitients

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Summary. We define the following functions: exponential function (for natural exponent), factorial function and Newton coefficients. We prove some basic properties of notions introduced. There is also a proof of binominal formula. We prove also that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

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The notation and terminology used in this paper have been introduced in the following articles: [4], [7], [6], [2], [3], [1], and [5]. We adopt the following rules: i, k, n, m, l denote natural numbers, a, b, x, y, z denote real numbers, and F, G denote finite sequences of elements of $\mathbb{R}$. One can prove the following propositions:

(1) For all x, y, z such that $y \neq 0$ and $z \neq 0$ holds $\frac{z \cdot x}{z \cdot y} = \frac{x}{y}$.

(2) If $k \geq l$, then $k - l$ is a natural number.

(3) For all F, G such that $\operatorname{len} F = \operatorname{len} G$ and for every i such that $i \in \operatorname{dom} F$ holds $F(i) = G(i)$ holds $F = G$.

(4) For every n such that $n \geq 1$ holds $1 \in \operatorname{Seg} n$.

(5) For every n such that $n \geq 1$ holds $\operatorname{Seg} n = (\{1\} \cup \{k : 1 < k \wedge k < n\}) \cup \{n\}$.

(6) For every F holds $\operatorname{len}(a \cdot F) = \operatorname{len} F$.

(7) $n \in \operatorname{dom} G$ if and only if $n \in \operatorname{dom}(a \cdot G)$.

Let us consider i, x. Then $i \longmapsto x$ is a finite sequence of elements of $\mathbb{R}$.

Let us consider x, n. The functor x^n yielding a real number is defined as follows:

(Def.1) $x^n = \prod(n \longmapsto x)$.

One can prove the following propositions:

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(8) $x^n = \prod(n \longmapsto x)$.

(9) For every x holds $x^0 = 1$.

(10) For every x holds $x^1 = x$.

(11) For every n holds $x^{n+1} = x^n \cdot x$ and $x^{n+1} = x \cdot x^n$.

(12) $(x \cdot y)^n = x^n \cdot y^n$.

(13) $x^{n+m} = x^n \cdot x^m$.

(14) $(x^n)^m = x^{n \cdot m}$.

(15) For every n holds $1^n = 1$.

(16) For every n such that $n \geq 1$ holds $0^n = 0$.

Let us consider n. Then id_n is a finite sequence of elements of $\mathbb{R}$.

Let us consider x. Then $\langle x \rangle$ is a finite sequence of elements of $\mathbb{R}$. Let us consider y. Then $\langle x, y \rangle$ is a finite sequence of elements of $\mathbb{R}$.

Let us consider n. The functor $n!$ yielding a real number is defined by:

(Def.2) $n! = \prod(\mathrm{id}_n)$.

We now state several propositions:

(17) $n! = \prod(\mathrm{id}_n)$.

(18) $0! = 1$.

(19) $1! = 1$.

(20) $2! = 2$.

(21) For every n holds $(n+1)! = (n+1) \cdot (n!)$ and $(n+1)! = (n!) \cdot (n+1)$.

(22) For every n holds $n!$ is a natural number.

(23) For every n holds $n! > 0$.

(24) For every n holds $n! \neq 0$.

(25) For all n, k holds $(n!) \cdot (k!) \neq 0$.

Let us consider k, n. The functor $\binom{n}{k}$ yielding a real number is defined as follows:

(Def.3) for every l such that $l = n - k$ holds $\binom{n}{k} = \frac{n!}{(k!) \cdot (l!)}$ if $n \geq k$, $\binom{n}{k} = 0$, otherwise.

We now state a number of propositions:

(26) For every l such that $l = n - k$ holds $\binom{n}{k} = \frac{n!}{(k!) \cdot (l!)}$ if and only if $n \geq k$ or if $\binom{n}{k} = 1$, then $n < k$.

(27) $\binom{0}{0} = 1$.

(28) For every k such that $k > 0$ holds $\binom{0}{k} = 0$.

(29) For every n holds $\binom{n}{0} = 1$.

(30) For all n, k such that $n \geq k$ for every l such that $l = n - k$ holds $\binom{n}{k} = \binom{n}{l}$.

(31) For every n holds $\binom{n}{n} = 1$.

(32) For all k, n such that $k < n$ holds $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$ and $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

(33) For every n such that $n \geq 1$ holds $\binom{n}{1} = n$.

(34) For all n, l such that $n \geq 1$ and $l = n - 1$ holds $\binom{n}{l} = n$.

(35) For every n and for every k holds $\binom{n}{k}$ is a natural number.

(36) For all m, F such that $m \neq 0$ and $\operatorname{len} F = m$ and for all i, l such that $i \in \operatorname{dom} F$ and $l = (n + i) - 1$ holds $F(i) = \binom{l}{n}$ holds $\sum F = \binom{n+m}{n+1}$.

Let a, b be real numbers, and let n be a natural number. The functor $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def.4) $\operatorname{len}\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle = n + 1$ and for all i, l, m such that $i \in \operatorname{dom}\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle$ and $m = i - 1$ and $l = n - m$ holds $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle(i) = (\binom{n}{m} \cdot a^l) \cdot b^m$.

Next we state several propositions:

(37) Given F. Then the following conditions are equivalent:

(i) $\operatorname{len} F = n + 1$ and for all i, l, m such that $i \in \operatorname{dom} F$ and $m = i - 1$ and $l = n - m$ holds $F(i) = (\binom{n}{m} \cdot a^l) \cdot b^m$,

(ii) $F = \langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle$.

(38) $\langle \binom{0}{0} a^0 b^0, \ldots, \binom{0}{0} a^0 b^0 \rangle = \langle 1 \rangle$.

(39) $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle(1) = a^n$.

(40) $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle(n + 1) = b^n$.

(41) For every n holds $(a + b)^n = \sum \langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle$.

Let us consider n. The functor $\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def.5) $\operatorname{len}\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle = n + 1$ and for all i, k such that $i \in \operatorname{dom}\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$ and $k = i - 1$ holds $\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle(i) = \binom{n}{k}$.

We now state three propositions:

(42) For every F holds $\operatorname{len} F = n + 1$ and for all i, m such that $i \in \operatorname{dom} F$ and $m = i - 1$ holds $F(i) = \binom{n}{m}$ if and only if $F = \langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$.

(43) For every n holds $\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle = \langle \binom{n}{0} 1^0 1^n, \ldots, \binom{n}{n} 1^n 1^0 \rangle$.

(44) For every n holds $2^n = \sum \langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$.

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