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Linear Combinations in Vector Space

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Summary. The notion of linear combination of vectors is introduced as a function from the carrier of a vector space to the carrier of the field. Definition of linear combination of set of vectors is also presented. We define addition and substraction of combinations and multiplication of combination by element of the field. Sum of finite set of vectors and sum of linear combination is defined. We prove theorems that belong rather to [5].

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The articles [12], [4], [2], [1], [3], [11], [7], [6], [9], [5], [8], and [10] provide the terminology and notation for this paper. Let D be a non-empty set. Then $\emptyset_D$ is a subset of D.

For simplicity we adopt the following rules: x will be arbitrary, i will be a natural number, G_1 will be a field, V will be a vector space over G_1, u, v, v_1, v_2, v_3 will be vectors of V, a, b, c will be elements of G_1, F, G will be finite sequences of elements of the carrier of the carrier of V, A, B will be subsets of V, and f will be a function from the carrier of the carrier of V into the carrier of G_1. Let us consider G_1, V. A subset of V is called a finite subset of V if:

(Def.1) it is finite.

We now state the proposition

(1) A is a finite subset of V if and only if A is finite.

In the sequel S, T are finite subsets of V. Let us consider G_1, V, S, T. Then $S \cup T$ is a finite subset of V. Then $S \cap T$ is a finite subset of V. Then $S \setminus T$ is a finite subset of V. Then $S \dot{-} T$ is a finite subset of V.

Let us consider G_1, V. The functor 0_V yields a finite subset of V and is defined as follows:

(Def.2) $0_V = \emptyset$.

One can prove the following proposition

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(2) $0_V = \emptyset$.

Let us consider G_1, V, T. The functor $\sum T$ yields a vector of V and is defined as follows:

(Def.3) there exists F such that $\operatorname{rng} F = T$ and F is one-to-one and $\sum T = \sum F$.

We now state two propositions:

(3) There exists F such that $\operatorname{rng} F = T$ and F is one-to-one and $\sum T = \sum F$.

(4) If $\operatorname{rng} F = T$ and F is one-to-one and $v = \sum F$, then $v = \sum T$.

Let us consider G_1, V, v. Then $\{v\}$ is a finite subset of V.

Let us consider G_1, V, v_1, v_2. Then $\{v_1, v_2\}$ is a finite subset of V.

Let us consider G_1, V, v_1, v_2, v_3. Then $\{v_1, v_2, v_3\}$ is a finite subset of V.

One can prove the following propositions:

(5) $\sum(0_V) = \Theta_V$.

(6) $\sum\{v\} = v$.

(7) If $v_1 \neq v_2$, then $\sum\{v_1, v_2\} = v_1 + v_2$.

(8) If $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_1 \neq v_3$, then $\sum\{v_1, v_2, v_3\} = (v_1 + v_2) + v_3$.

(9) If T misses S, then $\sum(T \cup S) = \sum T + \sum S$.

(10) $\sum(T \cup S) = (\sum T + \sum S) - \sum(T \cap S)$.

(11) $\sum(T \cap S) = (\sum T + \sum S) - \sum(T \cup S)$.

(12) $\sum(T \setminus S) = \sum(T \cup S) - \sum S$.

(13) $\sum(T \setminus S) = \sum T - \sum(T \cap S)$.

(14) $\sum(T \dot{-} S) = \sum(T \cup S) - \sum(T \cap S)$.

(15) $\sum(T \dot{-} S) = \sum(T \setminus S) + \sum(S \setminus T)$.

Let us consider G_1, V. An element of $(\text{the carrier of } G_1)^{\text{the carrier of the carrier of } V}$ is called a linear combination of V if:

(Def.4) there exists T such that for every v such that $v \notin T$ holds $\mathrm{it}(v) = 0_{G_1}$.

In the sequel K, L, L_1, L_2, L_3 are linear combinations of V. Next we state the proposition

(16) There exists T such that for every v such that $v \notin T$ holds $L(v) = 0_{G_1}$.

In the sequel E is an element of $(\text{the carrier of } G_1)^{\text{the carrier of the carrier of } V}$. We now state the proposition

(17) If there exists T such that for every v such that $v \notin T$ holds $E(v) = 0_{G_1}$, then E is a linear combination of V.

Let us consider G_1, V, L. The functor $\operatorname{support} L$ yields a finite subset of V and is defined as follows:

(Def.5) $\operatorname{support} L = \{v : L(v) \neq 0_{G_1}\}$.

The following propositions are true:

(18) $\operatorname{support} L = \{v : L(v) \neq 0_{G_1}\}$.

(19) $x \in \operatorname{support} L$ if and only if there exists v such that $x = v$ and $L(v) \neq 0_{G_1}$.

(20) $L(v) = 0_{G_1}$ if and only if $v \notin \operatorname{support} L$.

Let us consider G_1, V. The functor $\mathbf{0}_{\mathrm{LC}_V}$ yielding a linear combination of V is defined as follows:

(Def.6) $\operatorname{support} \mathbf{0}_{\mathrm{LC}_V} = \emptyset$.

Next we state two propositions:

(21) $L = \mathbf{0}_{\mathrm{LC}_V}$ if and only if $\operatorname{support} L = \emptyset$.

(22) $\mathbf{0}_{\mathrm{LC}_V}(v) = 0_{G_1}$.

Let us consider G_1, V, A. A linear combination of V is said to be a linear combination of A if:

(Def.7) $\operatorname{support} \mathrm{it} \subseteq A$.

One can prove the following proposition

(23) If $\operatorname{support} L \subseteq A$, then L is a linear combination of A.

In the sequel l denotes a linear combination of A. Next we state several propositions:

(24) $\operatorname{support} l \subseteq A$.

(25) If $A \subseteq B$, then l is a linear combination of B.

(26) $\mathbf{0}_{\mathrm{LC}_V}$ is a linear combination of A.

(27) For every linear combination l of $\emptyset_{\text{the carrier of the carrier of } V}$ holds $l = \mathbf{0}_{\mathrm{LC}_V}$.

(28) L is a linear combination of $\operatorname{support} L$.

Let us consider G_1, V, F, f. The functor $f \cdot F$ yields a finite sequence of elements of the carrier of the carrier of V and is defined by:

(Def.8) $\operatorname{len}(f \cdot F) = \operatorname{len} F$ and for every i such that $i \in \operatorname{dom}(f \cdot F)$ holds $(f \cdot F)(i) = f(\pi_i F) \cdot \pi_i F$.

Next we state several propositions:

(29) $\operatorname{len}(f \cdot F) = \operatorname{len} F$.

(30) For every i such that $i \in \operatorname{dom}(f \cdot F)$ holds $(f \cdot F)(i) = f(\pi_i F) \cdot \pi_i F$.

(31) If $\operatorname{len} G = \operatorname{len} F$ and for every i such that $i \in \operatorname{dom} G$ holds $G(i) = f(\pi_i F) \cdot \pi_i F$, then $G = f \cdot F$.

(32) If $i \in \operatorname{dom} F$ and $v = F(i)$, then $(f \cdot F)(i) = f(v) \cdot v$.

(33) $f \cdot \varepsilon_{\text{the carrier of the carrier of } V} = \varepsilon_{\text{the carrier of the carrier of } V}$.

(34) $f \cdot \langle v \rangle = \langle f(v) \cdot v \rangle$.

(35) $f \cdot \langle v_1, v_2 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2 \rangle$.

(36) $f \cdot \langle v_1, v_2, v_3 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2, f(v_3) \cdot v_3 \rangle$.

(37) $f \cdot (F \frown G) = (f \cdot F) \frown (f \cdot G)$.

Let us consider G_1, V, L. The functor $\sum L$ yielding a vector of V is defined as follows:

(Def.9) there exists F such that F is one-to-one and $\operatorname{rng} F = \operatorname{support} L$ and $\sum L = \sum(L \cdot F)$.

The following propositions are true:

(38) There exists F such that F is one-to-one and $\operatorname{rng} F = \operatorname{support} L$ and $\sum L = \sum(L \cdot F)$.

(39) If F is one-to-one and $\operatorname{rng} F = \operatorname{support} L$ and $u = \sum(L \cdot F)$, then $u = \sum L$.

(40) $A \neq \emptyset$ and A is linearly closed if and only if for every l holds $\sum l \in A$.

(41) $\sum \mathbf{0}_{\mathrm{LC}_V} = \Theta_V$.

(42) For every linear combination l of $\emptyset_{\text{the carrier of the carrier of } V}$ holds $\sum l = \Theta_V$.

(43) For every linear combination l of $\{v\}$ holds $\sum l = l(v) \cdot v$.

(44) If $v_1 \neq v_2$, then for every linear combination l of $\{v_1, v_2\}$ holds $\sum l = l(v_1) \cdot v_1 + l(v_2) \cdot v_2$.

(45) If $\operatorname{support} L = \emptyset$, then $\sum L = \Theta_V$.

(46) If $\operatorname{support} L = \{v\}$, then $\sum L = L(v) \cdot v$.

(47) If $\operatorname{support} L = \{v_1, v_2\}$ and $v_1 \neq v_2$, then $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

Let us consider G_1, V, L_1, L_2. Let us note that one can characterize the predicate $L_1 = L_2$ by the following (equivalent) condition:

(Def.10) for every v holds $L_1(v) = L_2(v)$.

One can prove the following proposition

(48) If for every v holds $L_1(v) = L_2(v)$, then $L_1 = L_2$.

Let us consider G_1, V, L_1, L_2. The functor $L_1 + L_2$ yields a linear combination of V and is defined as follows:

(Def.11) for every v holds $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

Next we state several propositions:

(49) If for every v holds $L(v) = L_1(v) + L_2(v)$, then $L = L_1 + L_2$.

(50) $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

(51) $\operatorname{support}(L_1 + L_2) \subseteq \operatorname{support} L_1 \cup \operatorname{support} L_2$.

(52) If L_1 is a linear combination of A and L_2 is a linear combination of A, then $L_1 + L_2$ is a linear combination of A.

(53) $L_1 + L_2 = L_2 + L_1$.

(54) $L_1 + (L_2 + L_3) = (L_1 + L_2) + L_3$.

(55) $L + \mathbf{0}_{\mathrm{LC}_V} = L$ and $\mathbf{0}_{\mathrm{LC}_V} + L = L$.

Let us consider G_1, V, a, L. The functor $a \cdot L$ yielding a linear combination of V is defined by:

(Def.12) for every v holds $(a \cdot L)(v) = a \cdot L(v)$.

The following propositions are true:

(56) If for every v holds $K(v) = a \cdot L(v)$, then $K = a \cdot L$.

(57) $(a \cdot L)(v) = a \cdot L(v)$.

(58) If $a \neq 0_{G_1}$, then $\mathrm{support}(a \cdot L) = \mathrm{support}\, L$.

(59) $0_{G_1} \cdot L = \mathbf{0}_{\mathrm{LC}_V}$.

(60) If L is a linear combination of A, then $a \cdot L$ is a linear combination of A.

(61) $(a + b) \cdot L = a \cdot L + b \cdot L$.

(62) $a \cdot (L_1 + L_2) = a \cdot L_1 + a \cdot L_2$.

(63) $a \cdot (b \cdot L) = (a \cdot b) \cdot L$.

(64) $(1_{G_1}) \cdot L = L$.

Let us consider G_1, V, L. The functor $-L$ yields a linear combination of V and is defined by:

(Def.13) $-L = (-1_{G_1}) \cdot L$.

The following propositions are true:

(65) $-L = (-1_{G_1}) \cdot L$.

(66) $(-L)(v) = -L(v)$.

(67) If $L_1 + L_2 = \mathbf{0}_{\mathrm{LC}_V}$, then $L_2 = -L_1$.

(68) $\mathrm{support}(-L) = \mathrm{support}\, L$.

(69) If L is a linear combination of A, then $-L$ is a linear combination of A.

(70) $-(-L) = L$.

Let us consider G_1, V, L_1, L_2. The functor $L_1 - L_2$ yielding a linear combination of V is defined by:

(Def.14) $L_1 - L_2 = L_1 + (-L_2)$.

Next we state a number of propositions:

(71) $L_1 - L_2 = L_1 + (-L_2)$.

(72) $(L_1 - L_2)(v) = L_1(v) - L_2(v)$.

(73) $\mathrm{support}(L_1 - L_2) \subseteq \mathrm{support}\, L_1 \cup \mathrm{support}\, L_2$.

(74) If L_1 is a linear combination of A and L_2 is a linear combination of A, then $L_1 - L_2$ is a linear combination of A.

(75) $L - L = \mathbf{0}_{\mathrm{LC}_V}$.

(76) $\sum(L_1 + L_2) = \sum L_1 + \sum L_2$.

(77) $\sum(a \cdot L) = a \cdot \sum L$.

(78) $\sum(-L) = -\sum L$.

(79) $\sum(L_1 - L_2) = \sum L_1 - \sum L_2$.

(80) $(-1_{G_1}) \cdot a = -a$.

(81) $-1_{G_1} \neq 0_{G_1}$.

(82) $-a = 0_{G_1} - a$.

(83) $-a = -(1_{G_1}) \cdot a$.

(84) $(a - b) \cdot c = a \cdot c - b \cdot c$.

(85) If $a + b = 0_{G_1}$, then $b = -a$.

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