FORMALIZED MATHEMATICS
Vol.1, No.**5**, November–December 1990
Université Catholique de Louvain

Consequences of the Reflection Theorem

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Summary. Some consequences of the reflection theorem are discussed. To formulate them the notions of elementary equivalence and subsystems, and of models for a set of formulae are introduced. Besides, the concept of cofinality of a ordinal number with second one is used. The consequences of the reflection theorem (it is sometimes called the Scott-Scarpellini lemma) are: (i) If A_ξ is a transfinite sequence as in the reflection theorem (see [9]) and $A = \bigcup_{\xi \in On} A_\xi$, then there is an increasing and continuous mapping ϕ from On into On such that for every critical number κ the set A_κ is an elementary subsystem of A ($A_\kappa \prec A$). (ii) There is an increasing continuous mapping $\phi : On \to On$ such that $\mathbf{R}_\kappa \prec V$ for each of its critical numbers κ (V is the universal class and On is the class of all ordinals belonging to V). (iii) There are ordinal numbers α cofinal with ω for which $\mathbf{R}_\alpha$ are models of ZF set theory. (iv) For each set X from universe V there is a model of ZF M which belongs to V and has X as an element.

MML Identifier: `ZFREFLE1`.

The articles [18], [14], [15], [19], [17], [8], [13], [5], [6], [1], [11], [4], [2], [7], [12], [16], [3], [10], and [9] provide the terminology and notation for this paper. We follow a convention: H, S will be ZF-formulae, X, Y will be sets, and e, u will be arbitrary. Let M be a non-empty family of sets, and let F be a subset of WFF. The predicate $M \models F$ is defined by:

(Def.1) for every H such that $H \in F$ holds $M \models H$.

We now define two new predicates. Let M_1, M_2 be non-empty families of sets. The predicate $M_1 \equiv M_2$ is defined as follows:

(Def.2) for every H such that $\mathrm{Free}\, H = \emptyset$ holds $M_1 \models H$ if and only if $M_2 \models H$.

Let us notice that this predicate is reflexive and symmetric. The predicate $M_1 \prec M_2$ is defined as follows:

[1]Supported by RPBP III-24.C1.

ISSN 0777–4028

(Def.3) $M_1 \subseteq M_2$ and for every H and for every function v from VAR into M_1 holds $M_1, v \models H$ if and only if $M_2, M_2[v] \models H$.

Let us observe that the predicate introduced above is reflexive.

The set $\mathbf{Ax}_{\rm ZF}$ is defined by:

(Def.4) $e \in \mathbf{Ax}_{\rm ZF}$ if and only if $e \in$ WFF but $e =$ the axiom of extensionality or $e =$ the axiom of pairs or $e =$ the axiom of unions or $e =$ the axiom of infinity or $e =$ the axiom of power sets or there exists H such that $\{x_0, x_1, x_2\}$ misses Free H and $e =$ the axiom of substitution for H.

Let us note that it makes sense to consider the following constant. Then $\mathbf{Ax}_{\rm ZF}$ is a subset of WFF.

Let D be a non-empty set. Then $\emptyset_D$ is a subset of D.

For simplicity we follow a convention: M, M_1, M_2 will be non-empty families of sets, f will be a function, F, F_1, F_2 will be subsets of WFF, W will be a universal class, a, b will be ordinals of W, A, B, C will be ordinal numbers, L will be a transfinite sequence of non-empty sets from W, and p_1, x_1 will be transfinite sequences of ordinals of W. We now state a number of propositions:

(1) $M \models \emptyset_{\rm WFF}$.

(2) If $F_1 \subseteq F_2$ and $M \models F_2$, then $M \models F_1$.

(3) If $M \models F_1$ and $M \models F_2$, then $M \models F_1 \cup F_2$.

(4) If M is a model of ZF, then $M \models \mathbf{Ax}_{\rm ZF}$.

(5) If $M \models \mathbf{Ax}_{\rm ZF}$ and M is transitive, then M is a model of ZF.

(6) There exists S such that Free $S = \emptyset$ and for every M holds $M \models S$ if and only if $M \models H$.

(7) $M_1 \equiv M_2$ if and only if for every H holds $M_1 \models H$ if and only if $M_2 \models H$.

(8) $M_1 \equiv M_2$ if and only if for every F holds $M_1 \models F$ if and only if $M_2 \models F$.

(9) If $M_1 \prec M_2$, then $M_1 \equiv M_2$.

(10) If M_1 is a model of ZF and $M_1 \equiv M_2$ and M_2 is transitive, then M_2 is a model of ZF.

In this article we present several logical schemes. The scheme *NonUniqBoundFunc* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function f such that dom $f = \mathcal{A}$ and rng $f \subseteq \mathcal{B}$ and for every e such that $e \in \mathcal{A}$ holds $\mathcal{P}[e, f(e)]$

provided the following requirement is met:

- for every e such that $e \in \mathcal{A}$ there exists u such that $u \in \mathcal{B}$ and $\mathcal{P}[e, u]$.

The scheme *NonUniqFuncEx* deals with a set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function f such that dom $f = \mathcal{A}$ and for every e such that $e \in \mathcal{A}$ holds $\mathcal{P}[e, f(e)]$

provided the following condition is met:

- for every e such that $e \in \mathcal{A}$ there exists u such that $\mathcal{P}[e, u]$.

The following propositions are true:

(11) If $X \subseteq W$ and $\overline{\overline{X}} < \overline{\overline{W}}$, then $X \in W$.

(12) If $\operatorname{dom} f \in W$ and $\operatorname{rng} f \subseteq W$, then $\operatorname{rng} f \in W$.

(13) If $X \approx Y$ or $\overline{\overline{X}} = \overline{\overline{Y}}$, then $2^X \approx 2^Y$ and $\overline{\overline{2^X}} = \overline{\overline{2^Y}}$.

(14) Let D be a non-empty set. Let P_1 be a function from D into $(\operatorname{On} W)^{\operatorname{On} W}$. Suppose $\overline{\overline{D}} < \overline{\overline{W}}$ and for every x_1 such that $x_1 \in \operatorname{rng} P_1$ holds x_1 is increasing and x_1 is continuous. Then there exists p_1 such that p_1 is increasing and p_1 is continuous and $p_1(\mathbf{0}_W) = \mathbf{0}_W$ and for every a holds $p_1(\operatorname{succ} a) = \sup(\{p_1(a)\} \cup \operatorname{uncurry} P_1{}^\circ[\!: D, \{\operatorname{succ} a\} :\!])$ and for every a such that $a \neq \mathbf{0}_W$ and a is a limit ordinal number holds $p_1(a) = \sup(p_1 \restriction a)$.

(15) For every sequence p_1 of ordinal numbers such that p_1 is increasing holds $C + p_1$ is increasing.

(16) For every sequence x_1 of ordinal numbers holds $(C + x_1) \restriction A = C + x_1 \restriction A$.

(17) For every sequence p_1 of ordinal numbers such that p_1 is increasing and p_1 is continuous holds $C + p_1$ is continuous.

Let A, B be ordinal numbers. We say that A is cofinal with B if and only if:

(Def.5) there exists a sequence x_1 of ordinal numbers such that $\operatorname{dom} x_1 = B$ and $\operatorname{rng} x_1 \subseteq A$ and x_1 is increasing and $A = \sup x_1$.

Let us notice that the predicate defined above is reflexive.

In the sequel p_2 will be a sequence of ordinal numbers. We now state a number of propositions:

(18) If p_2 is increasing and $A \subseteq B$ and $B \in \operatorname{dom} p_2$, then $p_2(A) \subseteq p_2(B)$.

(19) If $e \in \operatorname{rng} p_2$, then e is an ordinal number.

(20) $\operatorname{rng} p_2 \subseteq \sup p_2$.

(21) If A is cofinal with B and B is cofinal with C, then A is cofinal with C.

(22) If A is cofinal with B, then $B \subseteq A$.

(23) If A is cofinal with B and B is cofinal with A, then $A = B$.

(24) If $\operatorname{dom} p_2 \neq \mathbf{0}$ and $\operatorname{dom} p_2$ is a limit ordinal number and p_2 is increasing and A is the limit of p_2, then A is cofinal with $\operatorname{dom} p_2$.

(25) $\operatorname{succ} A$ is cofinal with $\mathbf{1}$.

(26) If A is cofinal with $\operatorname{succ} B$, then there exists C such that $A = \operatorname{succ} C$.

(27) If A is cofinal with B, then A is a limit ordinal number if and only if B is a limit ordinal number.

(28) If A is cofinal with $\mathbf{0}$, then $A = \mathbf{0}$.

(29) $\operatorname{On} W$ is not cofinal with a.

(30) If $\omega \in W$ and p_1 is increasing and p_1 is continuous, then there exists b such that $a \in b$ and $p_1(b) = b$.

(31) If $\omega \in W$ and p_1 is increasing and p_1 is continuous, then there exists a such that $b \in a$ and $p_1(a) = a$ and a is cofinal with ω.

(32) Suppose $\omega \in W$ and for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$ and for every a such that $a \neq \mathbf{0}$ and a is a limit ordinal number holds $L(a) = \bigcup(L \restriction a)$. Then there exists p_1 such that p_1 is increasing and p_1 is continuous and for every a such that $p_1(a) = a$ and $\mathbf{0} \neq a$ holds $L(a) \prec \bigcup L$.

(33) $\mathbf{R}_a \in W$.

(34) If $a \neq \mathbf{0}$, then $\mathbf{R}_a$ is a non-empty set from W.

(35) If $\omega \in W$, then there exists p_1 such that p_1 is increasing and p_1 is continuous and for all a, M such that $p_1(a) = a$ and $\mathbf{0} \neq a$ and $M = \mathbf{R}_a$ holds $M \prec W$.

(36) If $\omega \in W$, then there exist b, M such that $a \in b$ and $M = \mathbf{R}_b$ and $M \prec W$.

(37) If $\omega \in W$, then there exist a, M such that a is cofinal with ω and $M = \mathbf{R}_a$ and $M \prec W$.

(38) Suppose $\omega \in W$ and for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$ and for every a such that $a \neq \mathbf{0}$ and a is a limit ordinal number holds $L(a) = \bigcup(L \restriction a)$. Then there exists p_1 such that p_1 is increasing and p_1 is continuous and for every a such that $p_1(a) = a$ and $\mathbf{0} \neq a$ holds $L(a) \equiv \bigcup L$.

(39) If $\omega \in W$, then there exists p_1 such that p_1 is increasing and p_1 is continuous and for all a, M such that $p_1(a) = a$ and $\mathbf{0} \neq a$ and $M = \mathbf{R}_a$ holds $M \equiv W$.

(40) If $\omega \in W$, then there exist b, M such that $a \in b$ and $M = \mathbf{R}_b$ and $M \equiv W$.

(41) If $\omega \in W$, then there exist a, M such that a is cofinal with ω and $M = \mathbf{R}_a$ and $M \equiv W$.

(42) If $\omega \in W$, then there exist a, M such that a is cofinal with ω and $M = \mathbf{R}_a$ and M is a model of ZF.

(43) If $\omega \in W$ and $X \in W$, then there exists M such that $X \in M$ and $M \in W$ and M is a model of ZF.

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Received August 13, 1990