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Complex Spaces [1]

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Summary. We introduce the concept of n-dimensional complex space. We prove a number of simple but useful theorems concerning addition, multiplication by scalars and similar basic concepts. We introduce metric and topology. We prove that an n-dimensional complex space is a Hausdorf space and that it is regular.

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The articles [20], [16], [12], [1], [21], [5], [22], [7], [8], [3], [17], [11], [2], [18], [19], [6], [4], [9], [10], [15], [14], and [13] provide the notation and terminology for this paper. We follow the rules: k, n will be natural numbers, r, r', r_1 will be real numbers, and c, c', c_1, c_2 will be elements of $\mathbb{C}$. In this article we present several logical schemes. The scheme *FuncDefUniq* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

for all functions f_1, f_2 from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f_1(x) = \mathcal{F}(x)$ and for every element x of $\mathcal{A}$ holds $f_2(x) = \mathcal{F}(x)$ holds $f_1 = f_2$
for all values of the parameters.

The scheme *UnOpDefuniq* deals with a non-empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$ and states that:

for all unary operations u_1, u_2 on $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $u_1(x) = \mathcal{F}(x)$ and for every element x of $\mathcal{A}$ holds $u_2(x) = \mathcal{F}(x)$ holds $u_1 = u_2$
for all values of the parameters.

The scheme *BinOpDefuniq* deals with a non-empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$ and states that:

for all binary operations o_1, o_2 on $\mathcal{A}$ such that for all elements a, b of $\mathcal{A}$ holds $o_1(a, b) = \mathcal{F}(a,b)$ and for all elements a, b of $\mathcal{A}$ holds $o_2(a, b) = \mathcal{F}(a,b)$ holds $o_1 = o_2$
for all values of the parameters.

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The binary operation $+_{\mathbb{C}}$ on $\mathbb{C}$ is defined as follows:

(Def.1) for all c_1, c_2 holds $+_{\mathbb{C}}(c_1, c_2) = c_1 + c_2$.

The following propositions are true:

(1) $+_{\mathbb{C}}$ is commutative.

(2) $+_{\mathbb{C}}$ is associative.

(3) $0_{\mathbb{C}}$ is a unity w.r.t. $+_{\mathbb{C}}$.

(4) $\mathbf{1}_{+_{\mathbb{C}}} = 0_{\mathbb{C}}$.

(5) $+_{\mathbb{C}}$ has a unity.

The unary operation $-_{\mathbb{C}}$ on $\mathbb{C}$ is defined as follows:

(Def.2) for every c holds $-_{\mathbb{C}}(c) = -c$.

Next we state three propositions:

(6) $-_{\mathbb{C}}$ is an inverse operation w.r.t. $+_{\mathbb{C}}$.

(7) $+_{\mathbb{C}}$ has an inverse operation.

(8) The inverse operation w.r.t. $+_{\mathbb{C}} = -_{\mathbb{C}}$.

The binary operation $-_{\mathbb{C}}$ on $\mathbb{C}$ is defined by:

(Def.3) $-_{\mathbb{C}} = +_{\mathbb{C}} \circ (\mathrm{id}_{\mathbb{C}}, -_{\mathbb{C}})$.

The following proposition is true

(9) $-_{\mathbb{C}}(c_1, c_2) = c_1 - c_2$.

The binary operation $\cdot_{\mathbb{C}}$ on $\mathbb{C}$ is defined by:

(Def.4) for all c_1, c_2 holds $\cdot_{\mathbb{C}}(c_1, c_2) = c_1 \cdot c_2$.

The following propositions are true:

(10) $\cdot_{\mathbb{C}}$ is commutative.

(11) $\cdot_{\mathbb{C}}$ is associative.

(12) $1_{\mathbb{C}}$ is a unity w.r.t. $\cdot_{\mathbb{C}}$.

(13) $\mathbf{1}_{\cdot_{\mathbb{C}}} = 1_{\mathbb{C}}$.

(14) $\cdot_{\mathbb{C}}$ has a unity.

(15) $\cdot_{\mathbb{C}}$ is distributive w.r.t. $+_{\mathbb{C}}$.

Let us consider c. The functor $\cdot_{\mathbb{C}}^{c}$ yields a unary operation on $\mathbb{C}$ and is defined by:

(Def.5) $\cdot_{\mathbb{C}}^{c} = \cdot_{\mathbb{C}}^{\circ}(c, \mathrm{id}_{\mathbb{C}})$.

We now state two propositions:

(16) $\cdot_{\mathbb{C}}^{c}(c') = c \cdot c'$.

(17) $\cdot_{\mathbb{C}}^{c}$ is distributive w.r.t. $+_{\mathbb{C}}$.

The function $|\cdot|_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{R}$ is defined by:

(Def.6) for every c holds $|\cdot|_{\mathbb{C}}(c) = |c|$.

In the sequel z, z_1, z_2 will be finite sequences of elements of $\mathbb{C}$. We now define two new functors. Let us consider z_1, z_2. The functor $z_1 + z_2$ yields a finite sequence of elements of $\mathbb{C}$ and is defined by:

(Def.7) $z_1 + z_2 = +^{\circ}_{\mathbb{C}}(z_1, z_2)$.

The functor $z_1 - z_2$ yielding a finite sequence of elements of $\mathbb{C}$ is defined as follows:

(Def.8) $z_1 - z_2 = -^{\circ}_{\mathbb{C}}(z_1, z_2)$.

Let us consider z. The functor $-z$ yielding a finite sequence of elements of $\mathbb{C}$ is defined by:

(Def.9) $-z = -_{\mathbb{C}} \cdot z$.

Let us consider c, z. The functor $c \cdot z$ yielding a finite sequence of elements of $\mathbb{C}$ is defined by:

(Def.10) $c \cdot z = \cdot^{c}_{\mathbb{C}} \cdot z$.

Let us consider z. The functor $|z|$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def.11) $|z| = |\cdot|_{\mathbb{C}} \cdot z$.

Let us consider n. The functor $\mathbb{C}^n$ yielding a non-empty set of finite sequences of $\mathbb{C}$ is defined by:

(Def.12) $\mathbb{C}^n = \mathbb{C}^n$.

We follow a convention: x, z, z_1, z_2, z_3 will denote elements of $\mathbb{C}^n$ and A, B will denote subsets of $\mathbb{C}^n$. One can prove the following propositions:

(18) $\operatorname{len} z = n$.

(19) For every element z of $\mathbb{C}^0$ holds $z = \varepsilon_{\mathbb{C}}$.

(20) $\varepsilon_{\mathbb{C}}$ is an element of $\mathbb{C}^0$.

(21) If $k \in \operatorname{Seg} n$, then $z(k) \in \mathbb{C}$.

(22) If $k \in \operatorname{Seg} n$, then $z(k)$ is an element of $\mathbb{C}$.

(23) If for every k such that $k \in \operatorname{Seg} n$ holds $z_1(k) = z_2(k)$, then $z_1 = z_2$.

Let us consider n, z_1, z_2. Then $z_1 + z_2$ is an element of $\mathbb{C}^n$.

Next we state three propositions:

(24) If $k \in \operatorname{Seg} n$ and $c_1 = z_1(k)$ and $c_2 = z_2(k)$, then $(z_1 + z_2)(k) = c_1 + c_2$.

(25) $z_1 + z_2 = z_2 + z_1$.

(26) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

Let us consider n. The functor $0^n_{\mathbb{C}}$ yielding a finite sequence of elements of $\mathbb{C}$ is defined by:

(Def.13) $0^n_{\mathbb{C}} = n \longmapsto 0_{\mathbb{C}}$.

Let us consider n. Then $0^n_{\mathbb{C}}$ is an element of $\mathbb{C}^n$.

Next we state two propositions:

(27) If $k \in \operatorname{Seg} n$, then $0^n_{\mathbb{C}}(k) = 0_{\mathbb{C}}$.

(28) $z + 0^n_{\mathbb{C}} = z$ and $z = 0^n_{\mathbb{C}} + z$.

Let us consider n, z. Then $-z$ is an element of $\mathbb{C}^n$.

Next we state several propositions:

(29) If $k \in \operatorname{Seg} n$ and $c = z(k)$, then $(-z)(k) = -c$.

(30) $z + (-z) = 0_{\mathbb{C}}^{n}$ and $(-z) + z = 0_{\mathbb{C}}^{n}$.

(31) If $z_1 + z_2 = 0_{\mathbb{C}}^{n}$, then $z_1 = -z_2$ and $z_2 = -z_1$.

(32) $-(-z) = z$.

(33) If $-z_1 = -z_2$, then $z_1 = z_2$.

(34) If $z_1 + z = z_2 + z$ or $z_1 + z = z + z_2$, then $z_1 = z_2$.

(35) $-(z_1 + z_2) = (-z_1) + (-z_2)$.

Let us consider n, z_1, z_2. Then $z_1 - z_2$ is an element of $\mathbb{C}^n$.

Next we state a number of propositions:

(36) If $k \in \operatorname{Seg} n$ and $c_1 = z_1(k)$ and $c_2 = z_2(k)$, then $(z_1 - z_2)(k) = c_1 - c_2$.

(37) $z_1 - z_2 = z_1 + (-z_2)$.

(38) $z - 0_{\mathbb{C}}^{n} = z$.

(39) $0_{\mathbb{C}}^{n} - z = -z$.

(40) $z_1 - (-z_2) = z_1 + z_2$.

(41) $-(z_1 - z_2) = z_2 - z_1$.

(42) $-(z_1 - z_2) = (-z_1) + z_2$.

(43) $z - z = 0_{\mathbb{C}}^{n}$.

(44) If $z_1 - z_2 = 0_{\mathbb{C}}^{n}$, then $z_1 = z_2$.

(45) $(z_1 - z_2) - z_3 = z_1 - (z_2 + z_3)$.

(46) $z_1 + (z_2 - z_3) = (z_1 + z_2) - z_3$.

(47) $z_1 - (z_2 - z_3) = (z_1 - z_2) + z_3$.

(48) $(z_1 - z_2) + z_3 = (z_1 + z_3) - z_2$.

(49) $z_1 = (z_1 + z) - z$.

(50) $z_1 + (z_2 - z_1) = z_2$.

(51) $z_1 = (z_1 - z) + z$.

Let us consider n, c, z. Then $c \cdot z$ is an element of $\mathbb{C}^n$.

One can prove the following propositions:

(52) If $k \in \operatorname{Seg} n$ and $c' = z(k)$, then $(c \cdot z)(k) = c \cdot c'$.

(53) $c_1 \cdot (c_2 \cdot z) = (c_1 \cdot c_2) \cdot z$.

(54) $(c_1 + c_2) \cdot z = c_1 \cdot z + c_2 \cdot z$.

(55) $c \cdot (z_1 + z_2) = c \cdot z_1 + c \cdot z_2$.

(56) $1_{\mathbb{C}} \cdot z = z$.

(57) $0_{\mathbb{C}} \cdot z = 0_{\mathbb{C}}^{n}$.

(58) $(-1_{\mathbb{C}}) \cdot z = -z$.

Let us consider n, z. Then $|z|$ is an element of $\mathbb{R}^n$.

Next we state four propositions:

(59) If $k \in \operatorname{Seg} n$ and $c = z(k)$, then $|z|(k) = |c|$.

(60) $|0_{\mathbb{C}}^{n}| = n \longmapsto 0$.

(61) $|-z| = |z|$.

(62) $|c \cdot z| = |c| \cdot |z|$.

Let z be a finite sequence of elements of $\mathbb{C}$. The functor $|z|$ yields a real number and is defined by:

(Def.14) $|z| = \sqrt{\sum({}^2|z|)}$.

One can prove the following propositions:

(63) $|0_{\mathbb{C}}^n| = 0$.

(64) If $|z| = 0$, then $z = 0_{\mathbb{C}}^n$.

(65) $0 \leq |z|$.

(66) $|-z| = |z|$.

(67) $|c \cdot z| = |c| \cdot |z|$.

(68) $|z_1 + z_2| \leq |z_1| + |z_2|$.

(69) $|z_1 - z_2| \leq |z_1| + |z_2|$.

(70) $|z_1| - |z_2| \leq |z_1 + z_2|$.

(71) $|z_1| - |z_2| \leq |z_1 - z_2|$.

(72) $|z_1 - z_2| = 0$ if and only if $z_1 = z_2$.

(73) If $z_1 \neq z_2$, then $0 < |z_1 - z_2|$.

(74) $|z_1 - z_2| = |z_2 - z_1|$.

(75) $|z_1 - z_2| \leq |z_1 - z| + |z - z_2|$.

Let us consider n, and let A be an element of $2^{\mathbb{C}^n}$. We say that A is open if and only if:

(Def.15) for every x such that $x \in A$ there exists r such that $0 < r$ and for every z such that $|z| < r$ holds $x + z \in A$.

Let us consider n, and let A be an element of $2^{\mathbb{C}^n}$. We say that A is closed if and only if:

(Def.16) for every x such that for every r such that $r > 0$ there exists z such that $|z| < r$ and $x + z \in A$ holds $x \in A$.

We now state four propositions:

(76) For every element A of $2^{\mathbb{C}^n}$ such that $A = \emptyset$ holds A is open.

(77) For every element A of $2^{\mathbb{C}^n}$ such that $A = \mathbb{C}^n$ holds A is open.

(78) For every family A_1 of subsets of $\mathbb{C}^n$ such that for every element A of $2^{\mathbb{C}^n}$ such that $A \in A_1$ holds A is open for every element A of $2^{\mathbb{C}^n}$ such that $A = \bigcup A_1$ holds A is open.

(79) For all subsets A, B of $\mathbb{C}^n$ such that A is open and B is open for every element C of $2^{\mathbb{C}^n}$ such that $C = A \cap B$ holds C is open.

Let us consider n, x, r. The functor $\mathrm{Ball}(x, r)$ yielding a subset of $\mathbb{C}^n$ is defined by:

(Def.17) $\mathrm{Ball}(x, r) = \{z : |z - x| < r\}$.

The following three propositions are true:

(80) $z \in \mathrm{Ball}(x, r)$ if and only if $|x - z| < r$.

(81) If $0 < r$, then $x \in \mathrm{Ball}(x, r)$.

(82) $\mathrm{Ball}(z_1, r_1)$ is open.

Now we present two schemes. The scheme *SubsetFD* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(x) : \mathcal{P}[x]\}$, where x is an element of $\mathcal{A}$, is a subset of $\mathcal{B}$

for all values of the parameters.

The scheme *SubsetFD2* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(x, y) : \mathcal{P}[x, y]\}$, where x is an element of $\mathcal{A}$, and y is an element of $\mathcal{B}$, is a subset of $\mathcal{C}$

for all values of the parameters.

Let us consider n, x, A. The functor $\rho(x, A)$ yielding a real number is defined by:

(Def.18) for every X being sets of real numbers such that $X = \{|x - z| : z \in A\}$ holds $\rho(x, A) = \inf X$.

Let us consider n, A, r. The functor $\mathrm{Ball}(A, r)$ yields a subset of $\mathbb{C}^n$ and is defined as follows:

(Def.19) $\mathrm{Ball}(A, r) = \{z : \rho(z, A) < r\}$.

Next we state a number of propositions:

(83) If for every r' such that $r' > 0$ holds $r + r' > r_1$, then $r \geq r_1$.

(84) For every X being sets of real numbers and for every r such that $X \neq \emptyset$ and for every r' such that $r' \in X$ holds $r \leq r'$ holds $\inf X \geq r$.

(85) If $A \neq \emptyset$, then $\rho(x, A) \geq 0$.

(86) If $A \neq \emptyset$, then $\rho(x + z, A) \leq \rho(x, A) + |z|$.

(87) If $x \in A$, then $\rho(x, A) = 0$.

(88) If $x \notin A$ and $A \neq \emptyset$ and A is closed, then $\rho(x, A) > 0$.

(89) If $A \neq \emptyset$, then $|z_1 - x| + \rho(x, A) \geq \rho(z_1, A)$.

(90) $z \in \mathrm{Ball}(A, r)$ if and only if $\rho(z, A) < r$.

(91) If $0 < r$ and $x \in A$, then $x \in \mathrm{Ball}(A, r)$.

(92) If $0 < r$, then $A \subseteq \mathrm{Ball}(A, r)$.

(93) If $A \neq \emptyset$, then $\mathrm{Ball}(A, r_1)$ is open.

Let us consider n, A, B. The functor $\rho(A, B)$ yields a real number and is defined as follows:

(Def.20) for every X being sets of real numbers such that $X = \{|x - z| : x \in A \wedge z \in B\}$ holds $\rho(A, B) = \inf X$.

Let X, Y be sets of real numbers. The functor $X + Y$ yields sets of real numbers and is defined as follows:

(Def.21) $X + Y = \{r + r_1 : r \in X \wedge r_1 \in Y\}$.

Next we state several propositions:

(94) For all X, Y being sets of real numbers such that $X \neq \emptyset$ and $Y \neq \emptyset$ holds $X + Y \neq \emptyset$.

(95) For all X, Y being sets of real numbers such that $X \neq \emptyset$ and X is lower bounded and $Y \neq \emptyset$ and Y is lower bounded holds $X + Y$ is lower bounded.

(96) For all X, Y being sets of real numbers such that $X \neq \emptyset$ and X is lower bounded and $Y \neq \emptyset$ and Y is lower bounded holds $\inf(X + Y) = \inf X + \inf Y$.

(97) For all X, Y being sets of real numbers such that Y is lower bounded and $X \neq \emptyset$ and for every r such that $r \in X$ there exists r_1 such that $r_1 \in Y$ and $r_1 \leq r$ holds $\inf X \geq \inf Y$.

(98) If $A \neq \emptyset$ and $B \neq \emptyset$, then $\rho(A, B) \geq 0$.

(99) $\rho(A, B) = \rho(B, A)$.

(100) If $A \neq \emptyset$ and $B \neq \emptyset$, then $\rho(x, A) + \rho(x, B) \geq \rho(A, B)$.

(101) If $A \cap B \neq \emptyset$, then $\rho(A, B) = 0$.

Let us consider n. The open subsets of $\mathbb{C}^n$ constitute a family of subsets of $\mathbb{C}^n$ defined by:

(Def.22) the open subsets of $\mathbb{C}^n = \{A : A \text{ is open }\}$, where A is an element of $2^{\mathbb{C}^n}$.

The following proposition is true

(102) For every element A of $2^{\mathbb{C}^n}$ holds $A \in$ the open subsets of $\mathbb{C}^n$ if and only if A is open.

Let us consider n. The n-dimensional complex space is a topological space defined by:

(Def.23) the n-dimensional complex space $= \langle \mathbb{C}^n$, the open subsets of $\mathbb{C}^n \rangle$.

We now state two propositions:

(103) The topology of the n-dimensional complex space = the open subsets of $\mathbb{C}^n$.

(104) The carrier of the n-dimensional complex space $= \mathbb{C}^n$.

In the sequel p denotes a point of the n-dimensional complex space and V denotes a subset of the n-dimensional complex space. Next we state several propositions:

(105) p is an element of $\mathbb{C}^n$.

(106) V is a subset of $\mathbb{C}^n$.

(107) For every subset A of $\mathbb{C}^n$ holds A is a subset of the n-dimensional complex space.

(108) For every subset A of $\mathbb{C}^n$ such that $A = V$ holds A is open if and only if V is open.

(109) For every subset A of $\mathbb{C}^n$ holds A is closed if and only if A^{c} is open.

(110) For every subset A of $\mathbb{C}^n$ such that $A = V$ holds A is closed if and only if V is closed.

(111) The n-dimensional complex space is a T_2 space.

(112) The n-dimensional complex space is a T_3 space.

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