FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Locally Connected Spaces

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Summary. This article is a continuation of [6]. We define a neighbourhood of a point and a neighbourhood of a set and prove some facts about them. Then the definitions of a locally connected space and a locally connected set are introduced. Some theorems on locally connected spaces are given (based on [5]). We also define a quasi-component of a point and prove some of its basic properties.

MML Identifier: `CONNSP_2`.

The papers [11], [10], [2], [13], [7], [1], [12], [9], [3], [8], [14], [6], and [4] provide the terminology and notation for this paper. Let X be a topological space, and let x be a point of X. A subset of X is called a neighborhood of x if:

(Def.1) $x \in \operatorname{Int} \mathrm{it}$.

Let X be a topological space, and let A be a subset of X. A subset of X is called a neighborhood of A if:

(Def.2) $A \subseteq \operatorname{Int} \mathrm{it}$.

In the sequel X will denote a topological space, x will denote a point of X, and A, U_1 will denote subsets of X. We now state a number of propositions:

(2)[2] A is a neighborhood of U_1 if and only if $U_1 \subseteq \operatorname{Int} A$.

(3) For every x and for all subsets A, B of X such that A is a neighborhood of x and B is a neighborhood of x holds $A \cup B$ is a neighborhood of x.

(4) For every x and for all subsets A, B of X holds A is a neighborhood of x and B is a neighborhood of x if and only if $A \cap B$ is a neighborhood of x.

(5) For every subset U_1 of X and for every point x of X such that U_1 is open and $x \in U_1$ holds U_1 is a neighborhood of x.

(6) For every subset U_1 of X and for every point x of X such that U_1 is a neighborhood of x holds $x \in U_1$.

[1] Supported by RPBP.III-24.C1

[2] The proposition (1) was either repeated or obvious.

ISSN 0777–4028

(7) For all U_1, x such that U_1 is a neighborhood of x there exists a subset V of X such that V is a neighborhood of x and V is open and $V \subseteq U_1$.

(8) For all U_1, x holds U_1 is a neighborhood of x if and only if there exists a subset V of X such that V is open and $V \subseteq U_1$ and $x \in V$.

(9) U_1 is open if and only if for every x such that $x \in U_1$ there exists a subset A of X such that A is a neighborhood of x and $A \subseteq U_1$.

(10) For every subset V of X holds V is a neighborhood of $\{x\}$ if and only if V is a neighborhood of x.

(11) For every subset B of X and for every point x of $X \restriction B$ and for every subset A of $X \restriction B$ and for every subset A_1 of X and for every point x_1 of X such that $B \neq \emptyset_X$ and B is open and A is a neighborhood of x and $A = A_1$ and $x = x_1$ holds A_1 is a neighborhood of x_1.

(12) For every subset B of X and for every point x of $X \restriction B$ and for every subset A of $X \restriction B$ and for every subset A_1 of X and for every point x_1 of X such that A_1 is a neighborhood of x_1 and $A = A_1$ and $x = x_1$ holds A is a neighborhood of x.

(13) For all subsets A, B of X such that A is a component of X and $A \subseteq B$ holds A is a component of B.

(14) For every subspace X_1 of X and for every point x of X and for every point x_1 of X_1 such that $x = x_1$ holds $\mathrm{Component}(x_1) \subseteq \mathrm{Component}(x)$.

Let X be a topological space, and let x be a point of X. We say that X is locally connected in x if and only if:

(Def.3) for every subset U_1 of X such that U_1 is a neighborhood of x there exists a subset V of X such that V is a neighborhood of x and V is connected and $V \subseteq U_1$.

Let X be a topological space. We say that X is locally connected if and only if:

(Def.4) for every point x of X holds X is locally connected in x.

Let X be a topological space, and let A be a subset of X, and let x be a point of X. We say that A is locally connected in x if and only if:

(Def.5) there exists a point x_1 of $X \restriction A$ such that $x_1 = x$ and $X \restriction A$ is locally connected in x_1.

The following proposition is true

(17)[3] A is locally connected in x if and only if there exists a point x_1 of $X \restriction A$ such that $x_1 = x$ and $X \restriction A$ is locally connected in x_1.

Let X be a topological space, and let A be a subset of X. We say that A is locally connected if and only if:

(Def.6) $X \restriction A$ is locally connected.

One can prove the following propositions:

[3]The propositions (15)–(16) were either repeated or obvious.

(19)[4] For every x holds X is locally connected in x if and only if for all subsets V, S of X such that V is a neighborhood of x and S is a component of V and $x \in S$ holds S is a neighborhood of x.

(20) For every x holds X is locally connected in x if and only if for every subset U_1 of X such that U_1 is open and $x \in U_1$ there exists a point x_1 of $X \restriction U_1$ such that $x_1 = x$ and $x \in \mathrm{Int\,Component}(x_1)$.

(21) If X is locally connected, then for every subset S of X such that S is a component of X holds S is open.

(22) If X is locally connected in x, then for every subset A of X such that A is open and $x \in A$ holds A is locally connected in x.

(23) If X is locally connected, then for every subset A of X such that $A \neq \emptyset_X$ and A is open holds A is locally connected.

(24) X is locally connected if and only if for all subsets A, B of X such that $A \neq \emptyset_X$ and A is open and B is a component of A holds B is open.

(25) If X is locally connected, then for every subset E of X and for every subset C of $X \restriction E$ such that $E \neq \emptyset_X$ and $C \neq \emptyset_{X \restriction E}$ and C is connected and C is open there exists a subset H of X such that H is open and H is connected and $C = E \cap H$.

(26) X is a T_4 space if and only if for all subsets A, C of X such that $A \neq \emptyset$ and $C \neq \Omega_X$ and $A \subseteq C$ and A is closed and C is open there exists a subset G of X such that G is open and $A \subseteq G$ and $\overline{G} \subseteq C$.

(27) Suppose X is locally connected and X is a T_4 space. Let A, B be subsets of X. Suppose $A \neq \emptyset$ and $B \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Then if A is connected and B is connected, then there exist subsets R, S of X such that R is connected and S is connected and R is open and S is open and $A \subseteq R$ and $B \subseteq S$ and $R \cap S = \emptyset$.

(28) For every point x of X and for every family F of subsets of X such that for every subset A of X holds $A \in F$ if and only if A is open closed and $x \in A$ holds $F \neq \emptyset$.

Let X be a topological space, and let x be a point of X. The quasi-component of x is a subset of X defined by:

(Def.7) there exists a family F of subsets of X such that for every subset A of X holds $A \in F$ if and only if A is open closed and $x \in A$ and $\bigcap F =$ the quasi-component of x.

We now state several propositions:

(29) $A =$ the quasi-component of x if and only if there exists a family F of subsets of X such that for every subset A of X holds $A \in F$ if and only if A is open closed and $x \in A$ and $\bigcap F = A$.

(30) $x \in$ the quasi-component of x.

[4]The proposition (18) was either repeated or obvious.

(31) If A is open closed and $x \in A$, then if $A \subseteq$ the quasi-component of x, then $A =$ the quasi-component of x.

(32) The quasi-component of x is closed.

(33) Component$(x) \subseteq$ the quasi-component of x.

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Received September 5, 1990