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The Limit of a Real Function at Infinity

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Summary. We introduce the halflines (*open* and *closed*), real sequences divergent to infinity (*plus* and *minus*) and the proper and improper limit of a real function at infinity. We prove basic properties of halflines, sequeces divergent to infinity and the limit of function at infinity.

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The articles [14], [4], [1], [2], [12], [10], [5], [6], [11], [15], [3], [7], [8], [13], and [9] provide the terminology and notation for this paper. For simplicity we follow a convention: r, r_1, r_2, g, g_1, g_2 are real numbers, X is a subset of $\mathbb{R}$, n, m, k are natural numbers, s_1, s_2, s_3 are sequences of real numbers, and f, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$. Let us consider n, m. Then $\max(n, m)$ is a natural number.

We now state four propositions:

(1) If $0 \leq r_1$ and $r_1 < r_2$ and $0 < g_1$ and $g_1 \leq g_2$, then $r_1 \cdot g_1 < r_2 \cdot g_2$.

(2) If $r \neq 0$, then $(-r)^{-1} = -r^{-1}$.

(3) If $r_1 < r_2$ and $r_2 < 0$ and $0 < g$, then $\frac{g}{r_2} < \frac{g}{r_1}$.

(4) If $r < 0$, then $r^{-1} < 0$.

Let us consider r. We introduce the functor $]-\infty, r[$ as a synonym of $\mathrm{HL}(r)$.

We now define three new functors. Let us consider r. The functor $]-\infty, r]$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def.1) $]-\infty, r] = \{g : g \leq r\}$.

The functor $[r, +\infty[$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def.2) $[r, +\infty[= \{g : r \leq g\}$.

The functor $]r, +\infty[$ yielding a subset of $\mathbb{R}$ is defined by:

(Def.3) $]r, +\infty[= \{g : r < g\}$.

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One can prove the following propositions:

(5) $X =]-\infty, r]$ if and only if $X = \{g : g \leq r\}$.

(6) $X = [r, +\infty[$ if and only if $X = \{g : r \leq g\}$.

(7) $X =]r, +\infty[$ if and only if $X = \{g : r < g\}$.

(8) If $r_1 \leq r_2$, then $]r_2, +\infty[\subseteq]r_1, +\infty[$.

(9) If $r_1 \leq r_2$, then $[r_2, +\infty[\subseteq [r_1, +\infty[$.

(10) $]r, +\infty[\subseteq [r, +\infty[$.

(11) $]r, g[\subseteq]r, +\infty[$.

(12) $[r, g] \subseteq [r, +\infty[$.

(13) If $r_1 \leq r_2$, then $]-\infty, r_1[\subseteq]-\infty, r_2[$.

(14) If $r_1 \leq r_2$, then $]-\infty, r_1] \subseteq]-\infty, r_2]$.

(15) $]-\infty, r[\subseteq]-\infty, r]$.

(16) $]g, r[\subseteq]-\infty, r[$.

(17) $[g, r] \subseteq]-\infty, r]$.

(18) $]-\infty, r[\cap]g, +\infty[=]g, r[$.

(19) $]-\infty, r] \cap [g, +\infty[= [g, r]$.

(20) If $r \leq r_1$, then $]r_1, r_2[\subseteq]r, +\infty[$ and $[r_1, r_2] \subseteq [r, +\infty[$.

(21) If $r < r_1$, then $[r_1, r_2] \subseteq]r, +\infty[$.

(22) If $r_2 \leq r$, then $]r_1, r_2[\subseteq]-\infty, r[$ and $[r_1, r_2] \subseteq]-\infty, r]$.

(23) If $r_2 < r$, then $[r_1, r_2] \subseteq]-\infty, r[$.

(24) $\mathbb{R} \setminus]r, +\infty[=]-\infty, r]$ and $\mathbb{R} \setminus [r, +\infty[=]-\infty, r[$ and $\mathbb{R} \setminus]-\infty, r[= [r, +\infty[$ and $\mathbb{R} \setminus]-\infty, r] =]r, +\infty[$.

(25) $\mathbb{R} \setminus]r_1, r_2[=]-\infty, r_1] \cup [r_2, +\infty[$ and $\mathbb{R} \setminus [r_1, r_2] =]-\infty, r_1[\cup]r_2, +\infty[$.

(26) If s_1 is non-decreasing, then s_1 is lower bounded but if s_1 is non-increasing, then s_1 is upper bounded.

(27) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and s_1 is non-decreasing, then for every n holds $s_1(n) < 0$.

(28) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and s_1 is non-increasing, then for every n holds $0 < s_1(n)$.

(29) If s_1 is convergent and $0 < \lim s_1$, then there exists n such that for every m such that $n \leq m$ holds $0 < s_1(m)$.

(30) If s_1 is convergent and $0 < \lim s_1$, then there exists n such that for every m such that $n \leq m$ holds $\frac{\lim s_1}{2} < s_1(m)$.

We now define two new predicates. Let us consider s_1. We say that s_1 is divergent to $+\infty$ if and only if:

(Def.4) for every r there exists n such that for every m such that $n \leq m$ holds $r < s_1(m)$.

We say that s_1 is divergent to $-\infty$ if and only if:

(Def.5) for every r there exists n such that for every m such that $n \leq m$ holds $s_1(m) < r$.

Next we state a number of propositions:

(33)[2] If s_1 is divergent to $+\infty$ or s_1 is divergent to $-\infty$, then there exists n such that for every m such that $n \leq m$ holds $s_1 \uparrow m$ is non-zero.

(34) If $s_1 \uparrow k$ is divergent to $+\infty$, then s_1 is divergent to $+\infty$ but if $s_1 \uparrow k$ is divergent to $-\infty$, then s_1 is divergent to $-\infty$.

(35) If s_2 is divergent to $+\infty$ and s_3 is divergent to $+\infty$, then $s_2 + s_3$ is divergent to $+\infty$.

(36) If s_2 is divergent to $+\infty$ and s_3 is lower bounded, then s_2+s_3 is divergent to $+\infty$.

(37) If s_2 is divergent to $+\infty$ and s_3 is divergent to $+\infty$, then $s_2 s_3$ is divergent to $+\infty$.

(38) If s_2 is divergent to $-\infty$ and s_3 is divergent to $-\infty$, then $s_2 + s_3$ is divergent to $-\infty$.

(39) If s_2 is divergent to $-\infty$ and s_3 is upper bounded, then $s_2 + s_3$ is divergent to $-\infty$.

(40) If s_1 is divergent to $+\infty$ and $r > 0$, then $r s_1$ is divergent to $+\infty$ but if s_1 is divergent to $+\infty$ and $r < 0$, then $r s_1$ is divergent to $-\infty$ but if s_1 is divergent to $+\infty$ and $r = 0$, then $\mathrm{rng}(r s_1) = \{0\}$ and $r s_1$ is constant.

(41) If s_1 is divergent to $-\infty$ and $r > 0$, then $r s_1$ is divergent to $-\infty$ but if s_1 is divergent to $-\infty$ and $r < 0$, then $r s_1$ is divergent to $+\infty$ but if s_1 is divergent to $-\infty$ and $r = 0$, then $\mathrm{rng}(r s_1) = \{0\}$ and $r s_1$ is constant.

(42) If s_1 is divergent to $+\infty$, then $-s_1$ is divergent to $-\infty$ but if s_1 is divergent to $-\infty$, then $-s_1$ is divergent to $+\infty$.

(43) If s_1 is lower bounded and s_2 is divergent to $-\infty$, then $s_1 - s_2$ is divergent to $+\infty$.

(44) If s_1 is upper bounded and s_2 is divergent to $+\infty$, then $s_1 - s_2$ is divergent to $-\infty$.

(45) If s_1 is divergent to $+\infty$ and s_2 is convergent, then $s_1 + s_2$ is divergent to $+\infty$.

(46) If s_1 is divergent to $-\infty$ and s_2 is convergent, then $s_1 + s_2$ is divergent to $-\infty$.

(47) If for every n holds $s_1(n) = n$, then s_1 is divergent to $+\infty$.

(48) If for every n holds $s_1(n) = -n$, then s_1 is divergent to $-\infty$.

(49) If s_2 is divergent to $+\infty$ and there exists r such that $r > 0$ and for every n holds $s_3(n) \geq r$, then $s_2 s_3$ is divergent to $+\infty$.

(50) If s_2 is divergent to $-\infty$ and there exists r such that $0 < r$ and for every n holds $s_3(n) \geq r$, then $s_2 s_3$ is divergent to $-\infty$.

(51) If s_2 is divergent to $-\infty$ and s_3 is divergent to $-\infty$, then $s_2 s_3$ is divergent to $+\infty$.

[2] The propositions (31)–(32) were either repeated or obvious.

(52) If s_1 is divergent to $+\infty$ or s_1 is divergent to $-\infty$, then $|s_1|$ is divergent to $+\infty$.

(53) If s_1 is divergent to $+\infty$ and s_2 is a subsequence of s_1, then s_2 is divergent to $+\infty$.

(54) If s_1 is divergent to $-\infty$ and s_2 is a subsequence of s_1, then s_2 is divergent to $-\infty$.

(55) If s_2 is divergent to $+\infty$ and s_3 is convergent and $0 < \lim s_3$, then $s_2 s_3$ is divergent to $+\infty$.

(56) If s_1 is non-decreasing and s_1 is not upper bounded, then s_1 is divergent to $+\infty$.

(57) If s_1 is non-increasing and s_1 is not lower bounded, then s_1 is divergent to $-\infty$.

(58) If s_1 is increasing and s_1 is not upper bounded, then s_1 is divergent to $+\infty$.

(59) If s_1 is decreasing and s_1 is not lower bounded, then s_1 is divergent to $-\infty$.

(60) If s_1 is monotone, then s_1 is convergent or s_1 is divergent to $+\infty$ or s_1 is divergent to $-\infty$.

(61) If s_1 is divergent to $+\infty$ or s_1 is divergent to $-\infty$ but s_1 is non-zero, then ${s_1}^{-1}$ is convergent and $\lim {s_1}^{-1} = 0$.

Next we state several propositions:

(62) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and there exists k such that for every n such that $k \leq n$ holds $0 < s_1(n)$, then ${s_1}^{-1}$ is divergent to $+\infty$.

(63) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and there exists k such that for every n such that $k \leq n$ holds $s_1(n) < 0$, then ${s_1}^{-1}$ is divergent to $-\infty$.

(64) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and s_1 is non-decreasing, then ${s_1}^{-1}$ is divergent to $-\infty$.

(65) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and s_1 is non-increasing, then ${s_1}^{-1}$ is divergent to $+\infty$.

(66) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and s_1 is increasing, then ${s_1}^{-1}$ is divergent to $-\infty$.

(67) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and s_1 is decreasing, then ${s_1}^{-1}$ is divergent to $+\infty$.

(68) If s_2 is bounded but s_3 is divergent to $+\infty$ or s_3 is divergent to $-\infty$ and s_3 is non-zero, then $\frac{s_2}{s_3}$ is convergent and $\lim \frac{s_2}{s_3} = 0$.

(69) If s_1 is divergent to $+\infty$ and for every n holds $s_1(n) \leq s_2(n)$, then s_2 is divergent to $+\infty$.

(70) If s_1 is divergent to $-\infty$ and for every n holds $s_2(n) \leq s_1(n)$, then s_2 is divergent to $-\infty$.

We now define several new predicates. Let us consider f. We say that f is convergent in $+\infty$ if and only if:

(Def.6) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and there exists g such that for every s_1 such that s_1 is divergent to $+\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

We say that f is divergent in $+\infty$ to $+\infty$ if and only if:

(Def.7) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is divergent to $+\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is divergent to $+\infty$.

We say that f is divergent in $+\infty$ to $-\infty$ if and only if:

(Def.8) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is divergent to $+\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is divergent to $-\infty$.

We say that f is convergent in $-\infty$ if and only if:

(Def.9) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and there exists g such that for every s_1 such that s_1 is divergent to $-\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

We say that f is divergent in $-\infty$ to $+\infty$ if and only if:

(Def.10) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is divergent to $-\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is divergent to $+\infty$.

We say that f is divergent in $-\infty$ to $-\infty$ if and only if:

(Def.11) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is divergent to $-\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is divergent to $-\infty$.

We now state a number of propositions:

(77)[3] f is convergent in $+\infty$ if and only if for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and there exists g such that for every g_1 such that $0 < g_1$ there exists r such that for every r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(78) f is convergent in $-\infty$ if and only if for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and there exists g such that for every g_1 such that $0 < g_1$ there exists r such that for every r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(79) f is divergent in $+\infty$ to $+\infty$ if and only if for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and for every g there exists r such that for every r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $g < f(r_1)$.

(80) f is divergent in $+\infty$ to $-\infty$ if and only if for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and for every g there exists r such that for every r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < g$.

[3]The propositions (71)–(76) were either repeated or obvious.

(81) f is divergent in $-\infty$ to $+\infty$ if and only if for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and for every g there exists r such that for every r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $g < f(r_1)$.

(82) f is divergent in $-\infty$ to $-\infty$ if and only if for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and for every g there exists r such that for every r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < g$.

(83) If f_1 is divergent in $+\infty$ to $+\infty$ and f_2 is divergent in $+\infty$ to $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is divergent in $+\infty$ to $+\infty$ and $f_1 f_2$ is divergent in $+\infty$ to $+\infty$.

(84) If f_1 is divergent in $+\infty$ to $-\infty$ and f_2 is divergent in $+\infty$ to $-\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is divergent in $+\infty$ to $-\infty$ and $f_1 f_2$ is divergent in $+\infty$ to $+\infty$.

(85) If f_1 is divergent in $-\infty$ to $+\infty$ and f_2 is divergent in $-\infty$ to $+\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is divergent in $-\infty$ to $+\infty$ and $f_1 f_2$ is divergent in $-\infty$ to $+\infty$.

(86) If f_1 is divergent in $-\infty$ to $-\infty$ and f_2 is divergent in $-\infty$ to $-\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is divergent in $-\infty$ to $-\infty$ and $f_1 f_2$ is divergent in $-\infty$ to $+\infty$.

(87) If f_1 is divergent in $+\infty$ to $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists r such that f_2 is lower bounded on $]r, +\infty[$, then $f_1 + f_2$ is divergent in $+\infty$ to $+\infty$.

(88) If f_1 is divergent in $+\infty$ to $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom}(f_1 f_2)$ and there exist r, r_1 such that $0 < r$ and for every g such that $g \in \operatorname{dom} f_2 \cap]r_1, +\infty[$ holds $r \leq f_2(g)$, then $f_1 f_2$ is divergent in $+\infty$ to $+\infty$.

(89) If f_1 is divergent in $-\infty$ to $+\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists r such that f_2 is lower bounded on $]-\infty, r[$, then $f_1 + f_2$ is divergent in $-\infty$ to $+\infty$.

(90) If f_1 is divergent in $-\infty$ to $+\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(f_1 f_2)$ and there exist r, r_1 such that $0 < r$ and for every g such that $g \in \operatorname{dom} f_2 \cap]-\infty, r_1[$ holds $r \leq f_2(g)$, then $f_1 f_2$ is divergent in $-\infty$ to $+\infty$.

(91) If f is divergent in $+\infty$ to $+\infty$ and $r > 0$, then rf is divergent in $+\infty$ to $+\infty$ but if f is divergent in $+\infty$ to $+\infty$ and $r < 0$, then rf is divergent in $+\infty$ to $-\infty$ but if f is divergent in $+\infty$ to $-\infty$ and $r > 0$, then rf is divergent in $+\infty$ to $-\infty$ but if f is divergent in $+\infty$ to $-\infty$ and $r < 0$, then rf is divergent in $+\infty$ to $+\infty$.

(92) If f is divergent in $-\infty$ to $+\infty$ and $r > 0$, then rf is divergent in $-\infty$ to $+\infty$ but if f is divergent in $-\infty$ to $+\infty$ and $r < 0$, then rf is divergent in $-\infty$ to $-\infty$ but if f is divergent in $-\infty$ to $-\infty$ and $r > 0$, then rf is divergent in $-\infty$ to $-\infty$ but if f is divergent in $-\infty$ to $-\infty$ and $r < 0$, then rf is divergent in $-\infty$ to $+\infty$.

(93) If f is divergent in $+\infty$ to $+\infty$ or f is divergent in $+\infty$ to $-\infty$, then

$|f|$ is divergent in $+\infty$ to $+\infty$.

(94) If f is divergent in $-\infty$ to $+\infty$ or f is divergent in $-\infty$ to $-\infty$, then $|f|$ is divergent in $-\infty$ to $+\infty$.

(95) If there exists r such that f is non-decreasing on $]r, +\infty[$ and f is not upper bounded on $]r, +\infty[$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$, then f is divergent in $+\infty$ to $+\infty$.

(96) If there exists r such that f is increasing on $]r, +\infty[$ and f is not upper bounded on $]r, +\infty[$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$, then f is divergent in $+\infty$ to $+\infty$.

(97) If there exists r such that f is non-increasing on $]r, +\infty[$ and f is not lower bounded on $]r, +\infty[$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$, then f is divergent in $+\infty$ to $-\infty$.

(98) If there exists r such that f is decreasing on $]r, +\infty[$ and f is not lower bounded on $]r, +\infty[$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$, then f is divergent in $+\infty$ to $-\infty$.

(99) If there exists r such that f is non-increasing on $]-\infty, r[$ and f is not upper bounded on $]-\infty, r[$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, f$, then f is divergent in $-\infty$ to $+\infty$.

(100) If there exists r such that f is decreasing on $]-\infty, r[$ and f is not upper bounded on $]-\infty, r[$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, f$, then f is divergent in $-\infty$ to $+\infty$.

(101) If there exists r such that f is non-decreasing on $]-\infty, r[$ and f is not lower bounded on $]-\infty, r[$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, f$, then f is divergent in $-\infty$ to $-\infty$.

The following propositions are true:

(102) If there exists r such that f is increasing on $]-\infty, r[$ and f is not lower bounded on $]-\infty, r[$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, f$, then f is divergent in $-\infty$ to $-\infty$.

(103) Suppose f_1 is divergent in $+\infty$ to $+\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$ and there exists r such that $\mathrm{dom}\, f \cap]r, +\infty[\subseteq \mathrm{dom}\, f_1 \cap]r, +\infty[$ and for every g such that $g \in \mathrm{dom}\, f \cap]r, +\infty[$ holds $f_1(g) \leq f(g)$. Then f is divergent in $+\infty$ to $+\infty$.

(104) Suppose f_1 is divergent in $+\infty$ to $-\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$ and there exists r such that $\mathrm{dom}\, f \cap]r, +\infty[\subseteq \mathrm{dom}\, f_1 \cap]r, +\infty[$ and for every g such that $g \in \mathrm{dom}\, f \cap]r, +\infty[$ holds $f(g) \leq f_1(g)$. Then f is divergent in $+\infty$ to $-\infty$.

(105) Suppose f_1 is divergent in $-\infty$ to $+\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, f$ and there exists r such that $\mathrm{dom}\, f \cap]-\infty, r[\subseteq \mathrm{dom}\, f_1 \cap]-\infty, r[$ and for every g such that $g \in \mathrm{dom}\, f \cap]-\infty, r[$ holds $f_1(g) \leq f(g)$. Then f is divergent in $-\infty$ to $+\infty$.

(106) Suppose f_1 is divergent in $-\infty$ to $-\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, f$ and there exists r such that $\mathrm{dom}\, f \cap]-\infty, r[\subseteq$

$\operatorname{dom} f_1 \cap]-\infty, r[$ and for every g such that $g \in \operatorname{dom} f \cap]-\infty, r[$ holds $f(g) \leq f_1(g)$. Then f is divergent in $-\infty$ to $-\infty$.

(107) If f_1 is divergent in $+\infty$ to $+\infty$ and there exists r such that $]r, +\infty[\subseteq \operatorname{dom} f \cap \operatorname{dom} f_1$ and for every g such that $g \in]r, +\infty[$ holds $f_1(g) \leq f(g)$, then f is divergent in $+\infty$ to $+\infty$.

(108) If f_1 is divergent in $+\infty$ to $-\infty$ and there exists r such that $]r, +\infty[\subseteq \operatorname{dom} f \cap \operatorname{dom} f_1$ and for every g such that $g \in]r, +\infty[$ holds $f(g) \leq f_1(g)$, then f is divergent in $+\infty$ to $-\infty$.

(109) If f_1 is divergent in $-\infty$ to $+\infty$ and there exists r such that $]-\infty, r[\subseteq \operatorname{dom} f \cap \operatorname{dom} f_1$ and for every g such that $g \in]-\infty, r[$ holds $f_1(g) \leq f(g)$, then f is divergent in $-\infty$ to $+\infty$.

(110) If f_1 is divergent in $-\infty$ to $-\infty$ and there exists r such that $]-\infty, r[\subseteq \operatorname{dom} f \cap \operatorname{dom} f_1$ and for every g such that $g \in]-\infty, r[$ holds $f(g) \leq f_1(g)$, then f is divergent in $-\infty$ to $-\infty$.

Let us consider f. Let us assume that f is convergent in $+\infty$. The functor $\lim_{+\infty} f$ yielding a real number is defined by:

(Def.12) for every s_1 such that s_1 is divergent to $+\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = \lim_{+\infty} f$.

Let us consider f. Let us assume that f is convergent in $-\infty$. The functor $\lim_{-\infty} f$ yields a real number and is defined by:

(Def.13) for every s_1 such that s_1 is divergent to $-\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = \lim_{-\infty} f$.

Next we state a number of propositions:

(111) If f is convergent in $+\infty$, then $\lim_{+\infty} f = g$ if and only if for every s_1 such that s_1 is divergent to $+\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(112) If f is convergent in $-\infty$, then $\lim_{-\infty} f = g$ if and only if for every s_1 such that s_1 is divergent to $-\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(113) If f is convergent in $-\infty$, then $\lim_{-\infty} f = g$ if and only if for every g_1 such that $0 < g_1$ there exists r such that for every r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(114) If f is convergent in $+\infty$, then $\lim_{+\infty} f = g$ if and only if for every g_1 such that $0 < g_1$ there exists r such that for every r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(115) If f is convergent in $+\infty$, then rf is convergent in $+\infty$ and $\lim_{+\infty}(rf) = r \cdot (\lim_{+\infty} f)$.

(116) If f is convergent in $+\infty$, then $-f$ is convergent in $+\infty$ and $\lim_{+\infty}(-f) = -\lim_{+\infty} f$.

(117) If f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom}(f_1 + f_2)$, then $f_1 + f_2$ is convergent in $+\infty$ and $\lim_{+\infty}(f_1 + f_2) = \lim_{+\infty} f_1 + \lim_{+\infty} f_2$.

(118) If f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom}(f_1 - f_2)$, then $f_1 - f_2$ is convergent in $+\infty$ and $\lim_{+\infty}(f_1 - f_2) = \lim_{+\infty} f_1 - \lim_{+\infty} f_2$.

(119) If f is convergent in $+\infty$ and $f^{-1}\{0\} = \emptyset$ and $\lim_{+\infty} f \neq 0$, then $\frac{1}{f}$ is convergent in $+\infty$ and $\lim_{+\infty} \frac{1}{f} = (\lim_{+\infty} f)^{-1}$.

(120) If f is convergent in $+\infty$, then $|f|$ is convergent in $+\infty$ and $\lim_{+\infty} |f| = |\lim_{+\infty} f|$.

(121) If f is convergent in $+\infty$ and $\lim_{+\infty} f \neq 0$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$, then $\frac{1}{f}$ is convergent in $+\infty$ and $\lim_{+\infty} \frac{1}{f} = (\lim_{+\infty} f)^{-1}$.

(122) If f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom}(f_1 f_2)$, then $f_1 f_2$ is convergent in $+\infty$ and $\lim_{+\infty}(f_1 f_2) = (\lim_{+\infty} f_1) \cdot (\lim_{+\infty} f_2)$.

(123) If f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and $\lim_{+\infty} f_2 \neq 0$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} \frac{f_1}{f_2}$, then $\frac{f_1}{f_2}$ is convergent in $+\infty$ and $\lim_{+\infty} \frac{f_1}{f_2} = \frac{\lim_{+\infty} f_1}{\lim_{+\infty} f_2}$.

(124) If f is convergent in $-\infty$, then rf is convergent in $-\infty$ and $\lim_{-\infty}(rf) = r \cdot (\lim_{-\infty} f)$.

(125) If f is convergent in $-\infty$, then $-f$ is convergent in $-\infty$ and $\lim_{-\infty}(-f) = -\lim_{-\infty} f$.

(126) If f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(f_1 + f_2)$, then $f_1 + f_2$ is convergent in $-\infty$ and $\lim_{-\infty}(f_1 + f_2) = \lim_{-\infty} f_1 + \lim_{-\infty} f_2$.

(127) If f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(f_1 - f_2)$, then $f_1 - f_2$ is convergent in $-\infty$ and $\lim_{-\infty}(f_1 - f_2) = \lim_{-\infty} f_1 - \lim_{-\infty} f_2$.

(128) If f is convergent in $-\infty$ and $f^{-1}\{0\} = \emptyset$ and $\lim_{-\infty} f \neq 0$, then $\frac{1}{f}$ is convergent in $-\infty$ and $\lim_{-\infty} \frac{1}{f} = (\lim_{-\infty} f)^{-1}$.

(129) If f is convergent in $-\infty$, then $|f|$ is convergent in $-\infty$ and $\lim_{-\infty} |f| = |\lim_{-\infty} f|$.

(130) If f is convergent in $-\infty$ and $\lim_{-\infty} f \neq 0$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$, then $\frac{1}{f}$ is convergent in $-\infty$ and $\lim_{-\infty} \frac{1}{f} = (\lim_{-\infty} f)^{-1}$.

(131) If f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(f_1 f_2)$, then $f_1 f_2$ is convergent in $-\infty$ and $\lim_{-\infty}(f_1 f_2) = (\lim_{-\infty} f_1) \cdot (\lim_{-\infty} f_2)$.

(132) If f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and $\lim_{-\infty} f_2 \neq 0$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} \frac{f_1}{f_2}$, then $\frac{f_1}{f_2}$ is convergent in $-\infty$ and $\lim_{-\infty} \frac{f_1}{f_2} = \frac{\lim_{-\infty} f_1}{\lim_{-\infty} f_2}$.

(133) If f_1 is convergent in $+\infty$ and $\lim_{+\infty} f_1 = 0$ and for every r there exists

g such that $r < g$ and $g \in \mathrm{dom}(f_1 f_2)$ and there exists r such that f_2 is bounded on $]r, +\infty[$, then $f_1 f_2$ is convergent in $+\infty$ and $\lim_{+\infty}(f_1 f_2) = 0$.

(134) If f_1 is convergent in $-\infty$ and $\lim_{-\infty} f_1 = 0$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_1 f_2)$ and there exists r such that f_2 is bounded on $]-\infty, r[$, then $f_1 f_2$ is convergent in $-\infty$ and $\lim_{-\infty}(f_1 f_2) = 0$.

(135) Suppose that
(i) f_1 is convergent in $+\infty$,
(ii) f_2 is convergent in $+\infty$,
(iii) $\lim_{+\infty} f_1 = \lim_{+\infty} f_2$,
(iv) for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$,
(v) there exists r such that $\mathrm{dom}\, f_1 \cap]r, +\infty[\subseteq \mathrm{dom}\, f_2 \cap]r, +\infty[$ and $\mathrm{dom}\, f \cap]r, +\infty[\subseteq \mathrm{dom}\, f_1 \cap]r, +\infty[$ or $\mathrm{dom}\, f_2 \cap]r, +\infty[\subseteq \mathrm{dom}\, f_1 \cap]r, +\infty[$ and $\mathrm{dom}\, f \cap]r, +\infty[\subseteq \mathrm{dom}\, f_2 \cap]r, +\infty[$ but for every g such that $g \in \mathrm{dom}\, f \cap]r, +\infty[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.

Then f is convergent in $+\infty$ and $\lim_{+\infty} f = \lim_{+\infty} f_1$.

(136) Suppose f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and $\lim_{+\infty} f_1 = \lim_{+\infty} f_2$ and there exists r such that $]r, +\infty[\subseteq (\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2) \cap \mathrm{dom}\, f$ and for every g such that $g \in]r, +\infty[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$. Then f is convergent in $+\infty$ and $\lim_{+\infty} f = \lim_{+\infty} f_1$.

(137) Suppose that
(i) f_1 is convergent in $-\infty$,
(ii) f_2 is convergent in $-\infty$,
(iii) $\lim_{-\infty} f_1 = \lim_{-\infty} f_2$,
(iv) for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, f$,
(v) there exists r such that $\mathrm{dom}\, f_1 \cap]-\infty, r[\subseteq \mathrm{dom}\, f_2 \cap]-\infty, r[$ and $\mathrm{dom}\, f \cap]-\infty, r[\subseteq \mathrm{dom}\, f_1 \cap]-\infty, r[$ or $\mathrm{dom}\, f_2 \cap]-\infty, r[\subseteq \mathrm{dom}\, f_1 \cap]-\infty, r[$ and $\mathrm{dom}\, f \cap]-\infty, r[\subseteq \mathrm{dom}\, f_2 \cap]-\infty, r[$ but for every g such that $g \in \mathrm{dom}\, f \cap]-\infty, r[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.

Then f is convergent in $-\infty$ and $\lim_{-\infty} f = \lim_{-\infty} f_1$.

(138) Suppose f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and $\lim_{-\infty} f_1 = \lim_{-\infty} f_2$ and there exists r such that $]-\infty, r[\subseteq (\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2) \cap \mathrm{dom}\, f$ and for every g such that $g \in]-\infty, r[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$. Then f is convergent in $-\infty$ and $\lim_{-\infty} f = \lim_{-\infty} f_1$.

(139) Suppose that
(i) f_1 is convergent in $+\infty$,
(ii) f_2 is convergent in $+\infty$,
(iii) there exists r such that $\mathrm{dom}\, f_1 \cap]r, +\infty[\subseteq \mathrm{dom}\, f_2 \cap]r, +\infty[$ and for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) \leq f_2(g)$ or $\mathrm{dom}\, f_2 \cap]r, +\infty[\subseteq \mathrm{dom}\, f_1 \cap]r, +\infty[$ and for every g such that $g \in \mathrm{dom}\, f_2 \cap]r, +\infty[$ holds $f_1(g) \leq f_2(g)$.

Then $\lim_{+\infty} f_1 \leq \lim_{+\infty} f_2$.

(140) Suppose that
(i) f_1 is convergent in $-\infty$,
(ii) f_2 is convergent in $-\infty$,

(iii) there exists r such that $\operatorname{dom} f_1 \cap]-\infty, r[\subseteq \operatorname{dom} f_2 \cap]-\infty, r[$ and for every g such that $g \in \operatorname{dom} f_1 \cap]-\infty, r[$ holds $f_1(g) \leq f_2(g)$ or $\operatorname{dom} f_2 \cap]-\infty, r[\subseteq \operatorname{dom} f_1 \cap]-\infty, r[$ and for every g such that $g \in \operatorname{dom} f_2 \cap]-\infty, r[$ holds $f_1(g) \leq f_2(g)$.

Then $\lim_{-\infty} f_1 \leq \lim_{-\infty} f_2$.

(141) If f is divergent in $+\infty$ to $+\infty$ or f is divergent in $+\infty$ to $-\infty$ but for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$, then $\frac{1}{f}$ is convergent in $+\infty$ and $\lim_{+\infty} \frac{1}{f} = 0$.

We now state several propositions:

(142) If f is divergent in $-\infty$ to $+\infty$ or f is divergent in $-\infty$ to $-\infty$ but for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$, then $\frac{1}{f}$ is convergent in $-\infty$ and $\lim_{-\infty} \frac{1}{f} = 0$.

(143) If f is convergent in $+\infty$ and $\lim_{+\infty} f = 0$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$ and there exists r such that for every g such that $g \in \operatorname{dom} f \cap]r, +\infty[$ holds $0 \leq f(g)$, then $\frac{1}{f}$ is divergent in $+\infty$ to $+\infty$.

(144) If f is convergent in $+\infty$ and $\lim_{+\infty} f = 0$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$ and there exists r such that for every g such that $g \in \operatorname{dom} f \cap]r, +\infty[$ holds $f(g) \leq 0$, then $\frac{1}{f}$ is divergent in $+\infty$ to $-\infty$.

(145) If f is convergent in $-\infty$ and $\lim_{-\infty} f = 0$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$ and there exists r such that for every g such that $g \in \operatorname{dom} f \cap]-\infty, r[$ holds $0 \leq f(g)$, then $\frac{1}{f}$ is divergent in $-\infty$ to $+\infty$.

(146) If f is convergent in $-\infty$ and $\lim_{-\infty} f = 0$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$ and there exists r such that for every g such that $g \in \operatorname{dom} f \cap]-\infty, r[$ holds $f(g) \leq 0$, then $\frac{1}{f}$ is divergent in $-\infty$ to $-\infty$.

(147) If f is convergent in $+\infty$ and $\lim_{+\infty} f = 0$ and there exists r such that for every g such that $g \in \operatorname{dom} f \cap]r, +\infty[$ holds $0 < f(g)$, then $\frac{1}{f}$ is divergent in $+\infty$ to $+\infty$.

(148) If f is convergent in $+\infty$ and $\lim_{+\infty} f = 0$ and there exists r such that for every g such that $g \in \operatorname{dom} f \cap]r, +\infty[$ holds $f(g) < 0$, then $\frac{1}{f}$ is divergent in $+\infty$ to $-\infty$.

(149) If f is convergent in $-\infty$ and $\lim_{-\infty} f = 0$ and there exists r such that for every g such that $g \in \operatorname{dom} f \cap]-\infty, r[$ holds $0 < f(g)$, then $\frac{1}{f}$ is divergent in $-\infty$ to $+\infty$.

(150) If f is convergent in $-\infty$ and $\lim_{-\infty} f = 0$ and there exists r such that for every g such that $g \in \operatorname{dom} f \cap]-\infty, r[$ holds $f(g) < 0$, then $\frac{1}{f}$ is divergent in $-\infty$ to $-\infty$.

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