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The Limit of a Composition of Real Functions

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Summary. The theorem on the proper and the improper limit of a composition of real functions at a point, at infinity and one-side limits at a point are presented.

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The terminology and notation used in this paper have been introduced in the following articles: [17], [4], [1], [2], [15], [13], [5], [8], [14], [16], [3], [10], [11], [12], [7], [9], and [6]. We follow a convention: r, r_1, r_2, g, g_1, g_2, x_0 will be real numbers and f_1, f_2 will be partial functions from $\mathbb{R}$ to $\mathbb{R}$. The following propositions are true:

(1) Let s be a sequence of real numbers. Then for every set X such that $\mathrm{rng}\, s \subseteq \mathrm{dom}(f_2 \cdot f_1) \cap X$ holds $\mathrm{rng}\, s \subseteq \mathrm{dom}(f_2 \cdot f_1)$ and $\mathrm{rng}\, s \subseteq X$ and $\mathrm{rng}\, s \subseteq \mathrm{dom}\, f_1$ and $\mathrm{rng}\, s \subseteq \mathrm{dom}\, f_1 \cap X$ and $\mathrm{rng}(f_1 \cdot s) \subseteq \mathrm{dom}\, f_2$.

(2) For every sequence of real numbers s and for every set X such that $\mathrm{rng}\, s \subseteq \mathrm{dom}(f_2 \cdot f_1) \setminus X$ holds $\mathrm{rng}\, s \subseteq \mathrm{dom}(f_2 \cdot f_1)$ and $\mathrm{rng}\, s \subseteq \mathrm{dom}\, f_1$ and $\mathrm{rng}\, s \subseteq \mathrm{dom}\, f_1 \setminus X$ and $\mathrm{rng}(f_1 \cdot s) \subseteq \mathrm{dom}\, f_2$.

(3) If f_1 is divergent in $+\infty$ to $+\infty$ and f_2 is divergent in $+\infty$ to $+\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $+\infty$.

(4) If f_1 is divergent in $+\infty$ to $+\infty$ and f_2 is divergent in $+\infty$ to $-\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $-\infty$.

(5) If f_1 is divergent in $+\infty$ to $-\infty$ and f_2 is divergent in $-\infty$ to $+\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $+\infty$.

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(6) If f_1 is divergent in $+\infty$ to $-\infty$ and f_2 is divergent in $-\infty$ to $-\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $-\infty$.

(7) If f_1 is divergent in $-\infty$ to $+\infty$ and f_2 is divergent in $+\infty$ to $+\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $+\infty$.

(8) If f_1 is divergent in $-\infty$ to $+\infty$ and f_2 is divergent in $+\infty$ to $-\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $-\infty$.

(9) If f_1 is divergent in $-\infty$ to $-\infty$ and f_2 is divergent in $-\infty$ to $+\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $+\infty$.

(10) If f_1 is divergent in $-\infty$ to $-\infty$ and f_2 is divergent in $-\infty$ to $-\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $-\infty$.

(11) If f_1 is left divergent to $+\infty$ in x_0 and f_2 is divergent in $+\infty$ to $+\infty$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is left divergent to $+\infty$ in x_0.

(12) If f_1 is left divergent to $+\infty$ in x_0 and f_2 is divergent in $+\infty$ to $-\infty$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is left divergent to $-\infty$ in x_0.

(13) If f_1 is left divergent to $-\infty$ in x_0 and f_2 is divergent in $-\infty$ to $+\infty$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is left divergent to $+\infty$ in x_0.

(14) If f_1 is left divergent to $-\infty$ in x_0 and f_2 is divergent in $-\infty$ to $-\infty$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is left divergent to $-\infty$ in x_0.

(15) If f_1 is right divergent to $+\infty$ in x_0 and f_2 is divergent in $+\infty$ to $+\infty$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is right divergent to $+\infty$ in x_0.

(16) If f_1 is right divergent to $+\infty$ in x_0 and f_2 is divergent in $+\infty$ to $-\infty$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is right divergent to $-\infty$ in x_0.

(17) If f_1 is right divergent to $-\infty$ in x_0 and f_2 is divergent in $-\infty$ to $+\infty$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is right divergent to $+\infty$ in x_0.

(18) If f_1 is right divergent to $-\infty$ in x_0 and f_2 is divergent in $-\infty$ to $-\infty$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is right divergent to $-\infty$ in x_0.

(19) Suppose that
  (i) f_1 is left convergent in x_0,
  (ii) f_2 is left divergent to $+\infty$ in $\lim_{x_0^-} f_1$,

(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap {]x_0 - g, x_0[}$ holds $f_1(r) < \lim_{x_0^-} f_1$.
Then $f_2 \cdot f_1$ is left divergent to $+\infty$ in x_0.

(20) Suppose that
(i) f_1 is left convergent in x_0,
(ii) f_2 is left divergent to $-\infty$ in $\lim_{x_0^-} f_1$,
(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap {]x_0 - g, x_0[}$ holds $f_1(r) < \lim_{x_0^-} f_1$.
Then $f_2 \cdot f_1$ is left divergent to $-\infty$ in x_0.

(21) Suppose that
(i) f_1 is left convergent in x_0,
(ii) f_2 is right divergent to $+\infty$ in $\lim_{x_0^-} f_1$,
(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap {]x_0 - g, x_0[}$ holds $\lim_{x_0^-} f_1 < f_1(r)$.
Then $f_2 \cdot f_1$ is left divergent to $+\infty$ in x_0.

(22) Suppose that
(i) f_1 is left convergent in x_0,
(ii) f_2 is right divergent to $-\infty$ in $\lim_{x_0^-} f_1$,
(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap {]x_0 - g, x_0[}$ holds $\lim_{x_0^-} f_1 < f_1(r)$.
Then $f_2 \cdot f_1$ is left divergent to $-\infty$ in x_0.

(23) Suppose that
(i) f_1 is right convergent in x_0,
(ii) f_2 is right divergent to $+\infty$ in $\lim_{x_0^+} f_1$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap {]x_0, x_0 + g[}$ holds $\lim_{x_0^+} f_1 < f_1(r)$.
Then $f_2 \cdot f_1$ is right divergent to $+\infty$ in x_0.

(24) Suppose that
(i) f_1 is right convergent in x_0,
(ii) f_2 is right divergent to $-\infty$ in $\lim_{x_0^+} f_1$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap {]x_0, x_0 + g[}$ holds $\lim_{x_0^+} f_1 < f_1(r)$.

Then $f_2 \cdot f_1$ is right divergent to $-\infty$ in x_0.

(25) Suppose that

(i) f_1 is right convergent in x_0,

(ii) f_2 is left divergent to $+\infty$ in $\lim_{x_0^+} f_1$,

(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $f_1(r) < \lim_{x_0^+} f_1$.

Then $f_2 \cdot f_1$ is right divergent to $+\infty$ in x_0.

(26) Suppose that

(i) f_1 is right convergent in x_0,

(ii) f_2 is left divergent to $-\infty$ in $\lim_{x_0^+} f_1$,

(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $f_1(r) < \lim_{x_0^+} f_1$.

Then $f_2 \cdot f_1$ is right divergent to $-\infty$ in x_0.

(27) If f_1 is convergent in $+\infty$ and f_2 is left divergent to $+\infty$ in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) < \lim_{+\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $+\infty$.

(28) If f_1 is convergent in $+\infty$ and f_2 is left divergent to $-\infty$ in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) < \lim_{+\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $-\infty$.

(29) If f_1 is convergent in $+\infty$ and f_2 is right divergent to $+\infty$ in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $\lim_{+\infty} f_1 < f_1(g)$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $+\infty$.

(30) If f_1 is convergent in $+\infty$ and f_2 is right divergent to $-\infty$ in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $\lim_{+\infty} f_1 < f_1(g)$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $-\infty$.

(31) If f_1 is convergent in $-\infty$ and f_2 is left divergent to $+\infty$ in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $f_1(g) < \lim_{-\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $+\infty$.

Next we state a number of propositions:

(32) If f_1 is convergent in $-\infty$ and f_2 is left divergent to $-\infty$ in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $f_1(g) < \lim_{-\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $-\infty$.

(33) If f_1 is convergent in $-\infty$ and f_2 is right divergent to $+\infty$ in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $\lim_{-\infty} f_1 < f_1(g)$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $+\infty$.

(34) If f_1 is convergent in $-\infty$ and f_2 is right divergent to $-\infty$ in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $\lim_{-\infty} f_1 < f_1(g)$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $-\infty$.

(35) Suppose f_1 is divergent to $+\infty$ in x_0 and f_2 is divergent in $+\infty$ to $+\infty$ and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$. Then $f_2 \cdot f_1$ is divergent to $+\infty$ in x_0.

(36) Suppose f_1 is divergent to $+\infty$ in x_0 and f_2 is divergent in $+\infty$ to $-\infty$ and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$. Then $f_2 \cdot f_1$ is divergent to $-\infty$ in x_0.

(37) Suppose f_1 is divergent to $-\infty$ in x_0 and f_2 is divergent in $-\infty$ to $+\infty$ and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$. Then $f_2 \cdot f_1$ is divergent to $+\infty$ in x_0.

(38) Suppose f_1 is divergent to $-\infty$ in x_0 and f_2 is divergent in $-\infty$ to $-\infty$ and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$. Then $f_2 \cdot f_1$ is divergent to $-\infty$ in x_0.

(39) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is divergent to $+\infty$ in $\lim_{x_0} f_1$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) \neq \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is divergent to $+\infty$ in x_0.

(40) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is divergent to $-\infty$ in $\lim_{x_0} f_1$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) \neq \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is divergent to $-\infty$ in x_0.

(41) Suppose that

(i) f_1 is convergent in x_0,
(ii) f_2 is right divergent to $+\infty$ in $\lim_{x_0} f_1$,
(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) > \lim_{x_0} f_1$.
Then $f_2 \cdot f_1$ is divergent to $+\infty$ in x_0.

(42) Suppose that
(i) f_1 is convergent in x_0,
(ii) f_2 is right divergent to $-\infty$ in $\lim_{x_0} f_1$,
(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) > \lim_{x_0} f_1$.
Then $f_2 \cdot f_1$ is divergent to $-\infty$ in x_0.

(43) Suppose that
(i) f_1 is right convergent in x_0,
(ii) f_2 is divergent to $+\infty$ in $\lim_{x_0^+} f_1$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $f_1(r) \neq \lim_{x_0^+} f_1$.
Then $f_2 \cdot f_1$ is right divergent to $+\infty$ in x_0.

(44) Suppose that
(i) f_1 is right convergent in x_0,
(ii) f_2 is divergent to $-\infty$ in $\lim_{x_0^+} f_1$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $f_1(r) \neq \lim_{x_0^+} f_1$.
Then $f_2 \cdot f_1$ is right divergent to $-\infty$ in x_0.

(45) If f_1 is convergent in $+\infty$ and f_2 is divergent to $+\infty$ in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) \neq \lim_{+\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $+\infty$.

(46) If f_1 is convergent in $+\infty$ and f_2 is divergent to $-\infty$ in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) \neq \lim_{+\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $-\infty$.

(47) If f_1 is convergent in $-\infty$ and f_2 is divergent to $+\infty$ in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and

there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $f_1(g) \neq \lim_{-\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $+\infty$.

(48) If f_1 is convergent in $-\infty$ and f_2 is divergent to $-\infty$ in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $f_1(g) \neq \lim_{-\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $-\infty$.

(49) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is left divergent to $+\infty$ in $\lim_{x_0} f_1$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) < \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is divergent to $+\infty$ in x_0.

(50) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is left divergent to $-\infty$ in $\lim_{x_0} f_1$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) < \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is divergent to $-\infty$ in x_0.

(51) Suppose that

(i) f_1 is left convergent in x_0,

(ii) f_2 is divergent to $+\infty$ in $\lim_{x_0^-} f_1$,

(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0 - g, x_0[$ holds $f_1(r) \neq \lim_{x_0^-} f_1$.

Then $f_2 \cdot f_1$ is left divergent to $+\infty$ in x_0.

(52) Suppose that

(i) f_1 is left convergent in x_0,

(ii) f_2 is divergent to $-\infty$ in $\lim_{x_0^-} f_1$,

(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0 - g, x_0[$ holds $f_1(r) \neq \lim_{x_0^-} f_1$.

Then $f_2 \cdot f_1$ is left divergent to $-\infty$ in x_0.

(53) If f_1 is divergent in $+\infty$ to $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is convergent in $+\infty$ and $\lim_{+\infty}(f_2 \cdot f_1) = \lim_{+\infty} f_2$.

(54) If f_1 is divergent in $+\infty$ to $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is convergent in $+\infty$ and $\lim_{+\infty}(f_2 \cdot f_1) = \lim_{-\infty} f_2$.

(55) If f_1 is divergent in $-\infty$ to $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is convergent in $-\infty$ and $\lim_{-\infty}(f_2 \cdot f_1) = \lim_{+\infty} f_2$.

(56) If f_1 is divergent in $-\infty$ to $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is convergent in $-\infty$ and $\lim_{-\infty}(f_2 \cdot f_1) = \lim_{-\infty} f_2$.

(57) If f_1 is left divergent to $+\infty$ in x_0 and f_2 is convergent in $+\infty$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is left convergent in x_0 and $\lim_{x_0^-}(f_2 \cdot f_1) = \lim_{+\infty} f_2$.

(58) If f_1 is left divergent to $-\infty$ in x_0 and f_2 is convergent in $-\infty$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is left convergent in x_0 and $\lim_{x_0^-}(f_2 \cdot f_1) = \lim_{-\infty} f_2$.

(59) If f_1 is right divergent to $+\infty$ in x_0 and f_2 is convergent in $+\infty$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is right convergent in x_0 and $\lim_{x_0^+}(f_2 \cdot f_1) = \lim_{+\infty} f_2$.

(60) If f_1 is right divergent to $-\infty$ in x_0 and f_2 is convergent in $-\infty$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is right convergent in x_0 and $\lim_{x_0^+}(f_2 \cdot f_1) = \lim_{-\infty} f_2$.

(61) Suppose that
 (i) f_1 is left convergent in x_0,
 (ii) f_2 is left convergent in $\lim_{x_0^-} f_1$,
 (iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
 (iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0 - g, x_0[$ holds $f_1(r) < \lim_{x_0^-} f_1$.

Then $f_2 \cdot f_1$ is left convergent in x_0 and $\lim_{x_0^-}(f_2 \cdot f_1) = \lim_{\lim_{x_0^-} f_1^-} f_2$.

(62) Suppose that
 (i) f_1 is right convergent in x_0,
 (ii) f_2 is right convergent in $\lim_{x_0^+} f_1$,
 (iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
 (iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $\lim_{x_0^+} f_1 < f_1(r)$.

Then $f_2 \cdot f_1$ is right convergent in x_0 and $\lim_{x_0^+}(f_2 \cdot f_1) = \lim_{\lim_{x_0^+} f_1^+} f_2$.

One can prove the following propositions:

(63) Suppose that
(i) f_1 is left convergent in x_0,
(ii) f_2 is right convergent in $\lim_{x_0^-} f_1$,
(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0 - g, x_0[$ holds $\lim_{x_0^-} f_1 < f_1(r)$.
Then $f_2 \cdot f_1$ is left convergent in x_0 and $\lim_{x_0^-}(f_2 \cdot f_1) = \lim_{\lim_{x_0^-} f_1^+} f_2$.

(64) Suppose that
(i) f_1 is right convergent in x_0,
(ii) f_2 is left convergent in $\lim_{x_0^+} f_1$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $f_1(r) < \lim_{x_0^+} f_1$.
Then $f_2 \cdot f_1$ is right convergent in x_0 and $\lim_{x_0^+}(f_2 \cdot f_1) = \lim_{\lim_{x_0^+} f_1^-} f_2$.

(65) Suppose f_1 is convergent in $+\infty$ and f_2 is left convergent in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) < \lim_{+\infty} f_1$. Then $f_2 \cdot f_1$ is convergent in $+\infty$ and $\lim_{+\infty}(f_2 \cdot f_1) = \lim_{\lim_{+\infty} f_1^-} f_2$.

(66) Suppose f_1 is convergent in $+\infty$ and f_2 is right convergent in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $\lim_{+\infty} f_1 < f_1(g)$. Then $f_2 \cdot f_1$ is convergent in $+\infty$ and $\lim_{+\infty}(f_2 \cdot f_1) = \lim_{\lim_{+\infty} f_1^+} f_2$.

(67) Suppose f_1 is convergent in $-\infty$ and f_2 is left convergent in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $f_1(g) < \lim_{-\infty} f_1$. Then $f_2 \cdot f_1$ is convergent in $-\infty$ and $\lim_{-\infty}(f_2 \cdot f_1) = \lim_{\lim_{-\infty} f_1^-} f_2$.

(68) Suppose f_1 is convergent in $-\infty$ and f_2 is right convergent in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $\lim_{-\infty} f_1 < f_1(g)$. Then $f_2 \cdot f_1$ is convergent in $-\infty$ and $\lim_{-\infty}(f_2 \cdot f_1) = \lim_{\lim_{-\infty} f_1^+} f_2$.

(69) Suppose f_1 is divergent to $+\infty$ in x_0 and f_2 is convergent in $+\infty$ and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$. Then $f_2 \cdot f_1$ is convergent in x_0 and $\lim_{x_0}(f_2 \cdot f_1) = \lim_{+\infty} f_2$.

(70) Suppose f_1 is divergent to $-\infty$ in x_0 and f_2 is convergent in $-\infty$ and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that

$r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$. Then $f_2 \cdot f_1$ is convergent in x_0 and $\lim_{x_0}(f_2 \cdot f_1) = \lim_{-\infty} f_2$.

(71) Suppose f_1 is convergent in $+\infty$ and f_2 is convergent in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) \neq \lim_{+\infty} f_1$. Then $f_2 \cdot f_1$ is convergent in $+\infty$ and $\lim_{+\infty}(f_2 \cdot f_1) = \lim_{\lim_{+\infty} f_1} f_2$.

(72) Suppose f_1 is convergent in $-\infty$ and f_2 is convergent in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $f_1(g) \neq \lim_{-\infty} f_1$. Then $f_2 \cdot f_1$ is convergent in $-\infty$ and $\lim_{-\infty}(f_2 \cdot f_1) = \lim_{\lim_{-\infty} f_1} f_2$.

(73) Suppose that
- (i) f_1 is convergent in x_0,
- (ii) f_2 is left convergent in $\lim_{x_0} f_1$,
- (iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,
- (iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) < \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is convergent in x_0 and $\lim_{x_0}(f_2 \cdot f_1) = \lim_{\lim_{x_0} f_1^-} f_2$.

(74) Suppose that
- (i) f_1 is left convergent in x_0,
- (ii) f_2 is convergent in $\lim_{x_0^-} f_1$,
- (iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
- (iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0 - g, x_0[$ holds $f_1(r) \neq \lim_{x_0^-} f_1$.

Then $f_2 \cdot f_1$ is left convergent in x_0 and $\lim_{x_0^-}(f_2 \cdot f_1) = \lim_{\lim_{x_0^-} f_1} f_2$.

(75) Suppose that
- (i) f_1 is convergent in x_0,
- (ii) f_2 is right convergent in $\lim_{x_0} f_1$,
- (iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,
- (iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $\lim_{x_0} f_1 < f_1(r)$.

Then $f_2 \cdot f_1$ is convergent in x_0 and $\lim_{x_0}(f_2 \cdot f_1) = \lim_{\lim_{x_0} f_1^+} f_2$.

(76) Suppose that
- (i) f_1 is right convergent in x_0,
- (ii) f_2 is convergent in $\lim_{x_0^+} f_1$,

(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $f_1(r) \neq \lim_{x_0^+} f_1$.

Then $f_2 \cdot f_1$ is right convergent in x_0 and $\lim_{x_0^+}(f_2 \cdot f_1) = \lim_{\lim_{x_0^+} f_1} f_2$.

(77) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is convergent in $\lim_{x_0} f_1$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) \neq \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is convergent in x_0 and $\lim_{x_0}(f_2 \cdot f_1) = \lim_{\lim_{x_0} f_1} f_2$.

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