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Integer and Rational Exponents

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Summary. The article includes definitions and theorems which are needed to define real exponent. The following notions are defined: natural exponent, integer exponent and rational exponent.

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The terminology and notation used in this paper are introduced in the following papers: [12], [15], [4], [10], [1], [2], [3], [9], [7], [8], [14], [11], [13], [6], and [5]. For simplicity we follow the rules: a, b, c will be real numbers, m, n will be natural numbers, k, l, i will be integers, p, q will be rational numbers, and s_1, s_2 will be sequences of real numbers. The following propositions are true:

(2)[2] If s_1 is convergent and for every n holds $s_1(n) \geq a$, then $\lim s_1 \geq a$.

(3) If s_1 is convergent and for every n holds $s_1(n) \leq a$, then $\lim s_1 \leq a$.

Let us consider a. The functor $(a^{\kappa})_{\kappa \in \mathbb{N}}$ yielding a sequence of real numbers is defined as follows:

(Def.1) $((a^{\kappa})_{\kappa \in \mathbb{N}})(0) = 1$ and for every m holds $((a^{\kappa})_{\kappa \in \mathbb{N}})(m+1) = ((a^{\kappa})_{\kappa \in \mathbb{N}})(m) \cdot a$.

Next we state two propositions:

(4) For every sequence of real numbers s and for every a holds $s = (a^{\kappa})_{\kappa \in \mathbb{N}}$ if and only if $s(0) = 1$ and for every m holds $s(m+1) = s(m) \cdot a$.

(5) For every a such that $a \neq 0$ for every m holds $(a^{\kappa})_{\kappa \in \mathbb{N}}(m) \neq 0$.

Let us consider a, n. The functor $a^n_{\mathbb{N}}$ yields a real number and is defined by:

(Def.2) $a^n_{\mathbb{N}} = (a^{\kappa})_{\kappa \in \mathbb{N}}(n)$.

Next we state a number of propositions:

(6) $a^n_{\mathbb{N}} = (a^{\kappa})_{\kappa \in \mathbb{N}}(n)$.

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[2]The proposition (1) was either repeated or obvious.

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(7) $a_{\mathbb{N}}^{n} \cdot a = a_{\mathbb{N}}^{n+1}$.

(8) $1_{\mathbb{N}}^{n} = 1$.

(9) $a_{\mathbb{N}}^{n+m} = a_{\mathbb{N}}^{n} \cdot a_{\mathbb{N}}^{m}$.

(10) $(a \cdot b)_{\mathbb{N}}^{n} = a_{\mathbb{N}}^{n} \cdot b_{\mathbb{N}}^{n}$.

(11) $a_{\mathbb{N}}^{n \cdot m} = (a_{\mathbb{N}}^{n})_{\mathbb{N}}^{m}$.

(12) If $0 \neq a$, then $0 \neq a_{\mathbb{N}}^{n}$.

(13) If $0 < a$, then $0 < a_{\mathbb{N}}^{n}$.

(14) If $a \neq 0$, then $\frac{1}{a}_{\mathbb{N}}^{n} = \frac{1}{a_{\mathbb{N}}^{n}}$.

(15) If $a \neq 0$, then $\frac{b}{a}_{\mathbb{N}}^{n} = \frac{b_{\mathbb{N}}^{n}}{a_{\mathbb{N}}^{n}}$.

(16) If $n \geq 1$, then $0_{\mathbb{N}}^{n} = 0$.

(17) If $0 < a$ and $a \leq b$, then $a_{\mathbb{N}}^{n} \leq b_{\mathbb{N}}^{n}$.

(18) If $0 \leq a$ and $a < b$ and $1 \leq n$, then $a_{\mathbb{N}}^{n} < b_{\mathbb{N}}^{n}$.

(19) If $a \geq 1$, then $a_{\mathbb{N}}^{n} \geq 1$.

(20) If $1 \leq a$ and $1 \leq n$, then $a \leq a_{\mathbb{N}}^{n}$.

(21) If $1 < a$ and $2 \leq n$, then $a < a_{\mathbb{N}}^{n}$.

(22) If $0 < a$ and $a \leq 1$ and $1 \leq n$, then $a_{\mathbb{N}}^{n} \leq a$.

(23) If $0 < a$ and $a < 1$ and $2 \leq n$, then $a_{\mathbb{N}}^{n} < a$.

(24) If $-1 < a$, then $(1 + a)_{\mathbb{N}}^{n} \geq 1 + n \cdot a$.

(25) If $0 < a$ and $a < 1$, then $(1 + a)_{\mathbb{N}}^{n} \leq 1 + 3_{\mathbb{N}}^{n} \cdot a$.

(26) If s_1 is convergent and for every n holds $s_2(n) = (s_1(n))_{\mathbb{N}}^{m}$, then s_2 is convergent and $\lim s_2 = (\lim s_1)_{\mathbb{N}}^{m}$.

Let us consider n, a. Let us assume that $1 \leq n$. The functor $\mathrm{root}_n(a)$ yields a real number and is defined as follows:

(Def.3) $(\mathrm{root}_n(a))_{\mathbb{N}}^{n} = a$ and $\mathrm{root}_n(a) > 0$ if $a > 0$, $\mathrm{root}_n(a) = 0$ if $a = 0$.

Next we state a number of propositions:

(27) For all a, b, n such that $1 \leq n$ holds if $a > 0$, then $b = \mathrm{root}_n(a)$ if and only if $b_{\mathbb{N}}^{n} = a$ and $b > 0$ but if $a = 0$, then $\mathrm{root}_n(a) = 0$.

(28) If $a \geq 0$ and $n \geq 1$, then $(\mathrm{root}_n(a))_{\mathbb{N}}^{n} = a$ and $\mathrm{root}_n(a_{\mathbb{N}}^{n}) = a$.

(29) If $n \geq 1$, then $\mathrm{root}_n(1) = 1$.

(30) If $a \geq 0$, then $\mathrm{root}_1(a) = a$.

(31) If $a \geq 0$ and $b \geq 0$ and $n \geq 1$, then $\mathrm{root}_n(a \cdot b) = \mathrm{root}_n(a) \cdot \mathrm{root}_n(b)$.

(32) If $a > 0$ and $n \geq 1$, then $\mathrm{root}_n(\frac{1}{a}) = \frac{1}{\mathrm{root}_n(a)}$.

(33) If $a \geq 0$ and $b > 0$ and $n \geq 1$, then $\mathrm{root}_n(\frac{a}{b}) = \frac{\mathrm{root}_n(a)}{\mathrm{root}_n(b)}$.

(34) If $a \geq 0$ and $n \geq 1$ and $m \geq 1$, then $\mathrm{root}_n(\mathrm{root}_m(a)) = \mathrm{root}_{n \cdot m}(a)$.

(35) If $a \geq 0$ and $n \geq 1$ and $m \geq 1$, then $\mathrm{root}_n(a) \cdot \mathrm{root}_m(a) = \mathrm{root}_{n \cdot m}(a_{\mathbb{N}}^{n+m})$.

(36) If $0 \leq a$ and $a \leq b$ and $n \geq 1$, then $\mathrm{root}_n(a) \leq \mathrm{root}_n(b)$.

(37) If $a \geq 0$ and $a < b$ and $n \geq 1$, then $\mathrm{root}_n(a) < \mathrm{root}_n(b)$.

(38) If $a \geq 1$ and $n \geq 1$, then $\mathrm{root}_n(a) \geq 1$ and $a \geq \mathrm{root}_n(a)$.

(39) If $0 \leq a$ and $a < 1$ and $n \geq 1$, then $a \leq \mathrm{root}_n(a)$ and $\mathrm{root}_n(a) < 1$.

(40) If $a > 0$ and $n \geq 1$, then $\mathrm{root}_n(a) - 1 \leq \frac{a-1}{n}$.

(41) If $a \geq 0$, then $\mathrm{root}_2(a) = \sqrt{a}$.

(42) For every sequence of real numbers s and for every a such that $a > 0$ and for every n such that $n \geq 1$ holds $s(n) = \mathrm{root}_n(a)$ holds s is convergent and $\lim s = 1$.

Let us consider a, k. Let us assume that $a \neq 0$. The functor $a^k_{\mathbb{Z}}$ yields a real number and is defined as follows:

(Def.4) $a^k_{\mathbb{Z}} = a^{|k|}_{\mathbb{N}}$ if $k \geq 0$, $a^k_{\mathbb{Z}} = (a^{|k|}_{\mathbb{N}})^{-1}$ if $k < 0$.

We now state a number of propositions:

(43) If $a \neq 0$, then if $k \geq 0$, then $a^k_{\mathbb{Z}} = a^{|k|}_{\mathbb{N}}$ but if $k < 0$, then $a^k_{\mathbb{Z}} = (a^{|k|}_{\mathbb{N}})^{-1}$.

(44) If $a \neq 0$, then for every i such that $i = 0$ holds $a^i_{\mathbb{Z}} = 1$.

(45) If $a \neq 0$, then for every i such that $i = 1$ holds $a^i_{\mathbb{Z}} = a$.

(46) If $a \neq 0$ and $i = n$, then $a^i_{\mathbb{Z}} = a^n_{\mathbb{N}}$.

(47) $1^k_{\mathbb{Z}} = 1$.

(48) If $a \neq 0$, then $a^k_{\mathbb{Z}} \neq 0$.

(49) If $a > 0$, then $a^k_{\mathbb{Z}} > 0$.

(50) If $a \neq 0$ and $b \neq 0$, then $(a \cdot b)^k_{\mathbb{Z}} = a^k_{\mathbb{Z}} \cdot b^k_{\mathbb{Z}}$.

(51) If $a \neq 0$, then $a^{-k}_{\mathbb{Z}} = \frac{1}{a^k_{\mathbb{Z}}}$.

(52) If $a \neq 0$, then $\frac{1}{a}^k_{\mathbb{Z}} = \frac{1}{a^k_{\mathbb{Z}}}$.

(53) If $a \neq 0$, then $a^{m-n}_{\mathbb{Z}} = \frac{a^m_{\mathbb{N}}}{a^n_{\mathbb{N}}}$.

(54) If $a \neq 0$, then $a^{k+l}_{\mathbb{Z}} = a^k_{\mathbb{Z}} \cdot a^l_{\mathbb{Z}}$.

(55) If $a \neq 0$, then $(a^k_{\mathbb{Z}})^l_{\mathbb{Z}} = a^{k \cdot l}_{\mathbb{Z}}$.

(56) If $a > 0$ and $n \geq 1$, then $(\mathrm{root}_n(a))^k_{\mathbb{Z}} = \mathrm{root}_n(a^k_{\mathbb{Z}})$.

Let us consider a, p. Let us assume that $a > 0$. The functor $a^p_{\mathbb{Q}}$ yielding a real number is defined by:

(Def.5) $a^p_{\mathbb{Q}} = \mathrm{root}_{\mathrm{den}\, p}(a^{\mathrm{num}\, p}_{\mathbb{Z}})$.

We now state a number of propositions:

(57) If $a > 0$, then $a^p_{\mathbb{Q}} = \mathrm{root}_{\mathrm{den}\, p}(a^{\mathrm{num}\, p}_{\mathbb{Z}})$.

(58) If $a > 0$ and $p = 0$, then $a^p_{\mathbb{Q}} = 1$.

(59) If $a > 0$ and $p = 1$, then $a^p_{\mathbb{Q}} = a$.

(60) If $a > 0$ and $p = n$, then $a^p_{\mathbb{Q}} = a^n_{\mathbb{N}}$.

(61) If $a > 0$ and $n \geq 1$ and $p = n^{-1}$, then $a^p_{\mathbb{Q}} = \mathrm{root}_n(a)$.

(62) $1^p_{\mathbb{Q}} = 1$.

(63) If $a > 0$, then $a^p_{\mathbb{Q}} > 0$.

(64) If $a > 0$, then $a^p_{\mathbb{Q}} \cdot a^q_{\mathbb{Q}} = a^{p+q}_{\mathbb{Q}}$.

(65) If $a > 0$, then $\frac{1}{a^p_{\mathbb{Q}}} = a^{-p}_{\mathbb{Q}}$.

(66) If $a > 0$, then $\frac{a_{\mathbb{Q}}^{p}}{a_{\mathbb{Q}}^{q}} = a_{\mathbb{Q}}^{p-q}$.

(67) If $a > 0$ and $b > 0$, then $(a \cdot b)_{\mathbb{Q}}^{p} = a_{\mathbb{Q}}^{p} \cdot b_{\mathbb{Q}}^{p}$.

(68) If $a > 0$, then $\frac{1}{a}_{\mathbb{Q}}^{p} = \frac{1}{a_{\mathbb{Q}}^{p}}$.

(69) If $a > 0$ and $b > 0$, then $\frac{a}{b}_{\mathbb{Q}}^{p} = \frac{a_{\mathbb{Q}}^{p}}{b_{\mathbb{Q}}^{p}}$.

(70) If $a > 0$, then $(a_{\mathbb{Q}}^{p})_{\mathbb{Q}}^{q} = a_{\mathbb{Q}}^{p \cdot q}$.

(71) If $a \geq 1$ and $p \geq 0$, then $a_{\mathbb{Q}}^{p} \geq 1$.

(72) If $a \geq 1$ and $p \leq 0$, then $a_{\mathbb{Q}}^{p} \leq 1$.

(73) If $a > 1$ and $p > 0$, then $a_{\mathbb{Q}}^{p} > 1$.

(74) If $a \geq 1$ and $p \geq q$, then $a_{\mathbb{Q}}^{p} \geq a_{\mathbb{Q}}^{q}$.

(75) If $a > 1$ and $p > q$, then $a_{\mathbb{Q}}^{p} > a_{\mathbb{Q}}^{q}$.

(76) If $a > 0$ and $a < 1$ and $p > 0$, then $a_{\mathbb{Q}}^{p} < 1$.

(77) If $a > 0$ and $a \leq 1$ and $p \leq 0$, then $a_{\mathbb{Q}}^{p} \geq 1$.

A sequence of real numbers is called a rational sequence if:

(Def.6) for every n holds it(n) is a rational number.

Let s be a rational sequence, and let us consider n. Then $s(n)$ is a rational number.

Next we state two propositions:

(79)[3] For every a there exists a rational sequence s such that s is convergent and $\lim s = a$ and for every n holds $s(n) \leq a$.

(80) For every a there exists a rational sequence s such that s is convergent and $\lim s = a$ and for every n holds $s(n) \geq a$.

Let us consider a, and let s be a rational sequence. Let us assume that $a > 0$. The functor $a_{\mathbb{Q}}^{s}$ yields a sequence of real numbers and is defined as follows:

(Def.7) for every n holds $(a_{\mathbb{Q}}^{s})(n) = a_{\mathbb{Q}}^{s(n)}$.

The following propositions are true:

(81) For every a and for every rational sequence s and for every s_1 such that $a > 0$ holds $s_1 = a_{\mathbb{Q}}^{s}$ if and only if for every n holds $s_1(n) = a_{\mathbb{Q}}^{s(n)}$.

(82) For every rational sequence s and for every a such that s is convergent and $a > 0$ holds $a_{\mathbb{Q}}^{s}$ is convergent.

(83) For all rational sequences s_1, s_2 and for every a such that s_1 is convergent and s_2 is convergent and $\lim s_1 = \lim s_2$ and $a > 0$ holds $a_{\mathbb{Q}}^{s_1}$ is convergent and $a_{\mathbb{Q}}^{s_2}$ is convergent and $\lim a_{\mathbb{Q}}^{s_1} = \lim a_{\mathbb{Q}}^{s_2}$.

Let us consider a, b. Let us assume that $a > 0$. The functor $a_{\mathbb{R}}^{b}$ yielding a real number is defined by:

(Def.8) there exists a rational sequence s such that s is convergent and $\lim s = b$ and $a_{\mathbb{Q}}^{s}$ is convergent and $\lim a_{\mathbb{Q}}^{s} = a_{\mathbb{R}}^{b}$.

We now state a number of propositions:

[3]The proposition (78) was either repeated or obvious.

(84) For all a, b, c such that $a > 0$ holds $c = a_{\mathbb{R}}^{b}$ if and only if there exists a rational sequence s such that s is convergent and $\lim s = b$ and $a_{\mathbb{Q}}^{s}$ is convergent and $\lim a_{\mathbb{Q}}^{s} = c$.

(85) If $a > 0$, then $a_{\mathbb{R}}^{0} = 1$.

(86) If $a > 0$, then $a_{\mathbb{R}}^{1} = a$.

(87) $1_{\mathbb{R}}^{a} = 1$.

(88) If $a > 0$, then $a_{\mathbb{R}}^{p} = a_{\mathbb{Q}}^{p}$.

(89) If $a > 0$, then $a_{\mathbb{R}}^{b+c} = a_{\mathbb{R}}^{b} \cdot a_{\mathbb{R}}^{c}$.

(90) If $a > 0$, then $a_{\mathbb{R}}^{-c} = \frac{1}{a_{\mathbb{R}}^{c}}$.

(91) If $a > 0$, then $a_{\mathbb{R}}^{b-c} = \frac{a_{\mathbb{R}}^{b}}{a_{\mathbb{R}}^{c}}$.

(92) If $a > 0$ and $b > 0$, then $(a \cdot b)_{\mathbb{R}}^{c} = a_{\mathbb{R}}^{c} \cdot b_{\mathbb{R}}^{c}$.

(93) If $a > 0$, then $\frac{1}{a}{}_{\mathbb{R}}^{c} = \frac{1}{a_{\mathbb{R}}^{c}}$.

(94) If $a > 0$ and $b > 0$, then $\frac{a}{b}{}_{\mathbb{R}}^{c} = \frac{a_{\mathbb{R}}^{c}}{b_{\mathbb{R}}^{c}}$.

(95) If $a > 0$, then $a_{\mathbb{R}}^{b} > 0$.

(96) If $a \geq 1$ and $c \geq b$, then $a_{\mathbb{R}}^{c} \geq a_{\mathbb{R}}^{b}$.

(97) If $a > 1$ and $c > b$, then $a_{\mathbb{R}}^{c} > a_{\mathbb{R}}^{b}$.

(98) If $a > 0$ and $a \leq 1$ and $c \geq b$, then $a_{\mathbb{R}}^{c} \leq a_{\mathbb{R}}^{b}$.

(99) If $a \geq 1$ and $b \geq 0$, then $a_{\mathbb{R}}^{b} \geq 1$.

(100) If $a > 1$ and $b > 0$, then $a_{\mathbb{R}}^{b} > 1$.

(101) If $a \geq 1$ and $b \leq 0$, then $a_{\mathbb{R}}^{b} \leq 1$.

(102) If $a > 1$ and $b < 0$, then $a_{\mathbb{R}}^{b} < 1$.

(103) If s_1 is convergent and s_2 is convergent and $\lim s_1 > 0$ and for every n holds $s_1(n) > 0$ and $s_2(n) = (s_1(n))_{\mathbb{Q}}^{p}$, then $\lim s_2 = (\lim s_1)_{\mathbb{Q}}^{p}$.

(104) If $a > 0$ and s_1 is convergent and s_2 is convergent and for every n holds $s_2(n) = a_{\mathbb{R}}^{s_1(n)}$, then $\lim s_2 = a_{\mathbb{R}}^{\lim s_1}$.

(105) If $a > 0$, then $(a_{\mathbb{R}}^{b})_{\mathbb{R}}^{c} = a_{\mathbb{R}}^{b \cdot c}$.

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