FORMALIZED MATHEMATICS
Vol.2,No.1, January–February 1991
Université Catholique de Louvain

Equalities and Inequalities in Real Numbers

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Summary. The article is to give a number of useful theorems concerning equalities and inequalities in real numbers. Some of the theorems are extentions of [1] theorems, others were found to be needed in practice.

MML Identifier: `REAL_2`.

The terminology and notation used here are introduced in the following articles: [1], [3], [2], and [4]. In the sequel a, b, d, e will be real numbers. One can prove the following propositions:

(1) If $b+a=b$ or $a+b=b$ or $b-a=b$, then $a=0$.

(2) Suppose that
(i) $a-b=0$ or $a+(-b)=0$ or $(-b)+a=0$ or $-a=-b$ or $a-e=b-e$ or $a-e=b+(-e)$ or $a-e=(-e)+b$ or $e-a=e-b$ or $e-a=e+(-b)$ or $e-a=(-b)+e$.
Then $a=b$.

(3) If $a=-b$, then $a+b=0$ and $b+a=0$ and $-a=b$.

(4) If $a+b=0$ or $b+a=0$, then $a=-b$.

(5) $(-a)-b=(-b)-a$.

(6) $-(a+b)=(-a)+(-b)$ and $-(a+b)=(-b)+(-a)$ and $-(a+b)=(-b)-a$ and $-(a+b)=(-a)-b$.

(7) $a-b=(-b)+a$.

(8) $-(a-b)=(-a)+b$ and $-(a-b)=b-a$ and $-(a-b)=b+(-a)$.

(9) $-((-a)+b)=a-b$ and $-((-a)+b)=a+(-b)$ and $-((-a)+b)=(-b)+a$.

(10) (i) $a+b=-((-a)-b)$,
(ii) $a+b=-((-b)-a)$,
(iii) $a+b=-((-b)+(-a))$,

ISSN 0777–4028

(iv) $a + b = -((-a) + (-b))$,
(v) $a + b = a - (-b)$,
(vi) $a + b = b - (-a)$.

(11) If $a+b = e$ or $b+a = e$, then $a = e-b$ and $a = e+(-b)$ and $a = (-b)+e$.

(12) If $a = e-b$ or $a = e+(-b)$ or $a = (-b)+e$, then $a+b = e$ and $b+a = e$ and $b = e - a$.

(13) If $a + b = e + d$, then $a - e = d - b$ and $a - d = e - b$ and $b - e = d - a$ and $b - d = e - a$.

(14) If $a - e = d - b$, then $a + b = e + d$ and $a + b = d + e$ and $b + a = d + e$ and $b + a = e + d$.

(15) If $a - b = e - d$, then $a - e = b - d$.

(16) If $a + b = e - d$ or $b + a = e - d$, then $a + d = e - b$ and $d + a = e - b$.

(17) (i) $a = a + (b - b)$,
(ii) $a = (a + b) - b$,
(iii) $a = a + (b + (-b))$,
(iv) $a = (a + b) + (-b)$,
(v) $a = a - (b - b)$,
(vi) $a = (a - b) + b$,
(vii) $a = a - (b + (-b))$,
(viii) $a = a + ((-b) + b)$,
(ix) $a = (a + (-b)) + b$,
(x) $a = b + (a - b)$,
(xi) $a = (b + a) - b$,
(xii) $a = b + (a + (-b))$,
(xiii) $a = (b + a) + (-b)$,
(xiv) $a = b - (b - a)$,
(xv) $a = (b - b) + a$,
(xvi) $a = (-b) + (a + b)$,
(xvii) $a = ((-b) + a) + b$,
(xviii) $a = (-b) + (b + a)$,
(xix) $a = ((-b) + b) + a$,
(xx) $a = (-b) - ((-a) - b)$,
(xxi) $a = (-b) - ((-b) - a)$.

(18) $a - (b - e) = a + (e - b)$ and $a + (b - e) = (a + b) - e$.

(20)[1] $a + ((-b) - e) = (a - b) - e$ and $a - ((-b) - e) = (a + b) + e$.

(21) $(a+b)+e = (a+e)+b$ and $(a+b)+e = (b+e)+a$ and $(a+b)+e = (e+a)+b$ and $(a + b) + e = (e + b) + a$.

(22) $(a + b) - e = (a - e) + b$ and $(a + b) - e = (b - e) + a$ and $(a + b) - e = ((-e) + a) + b$ and $(a + b) - e = ((-e) + b) + a$.

(23) $(a-b)+e = (e-b)+a$ and $(a-b)+e = ((-b)+a)+e$ and $(a-b)+e = ((-b) + e) + a$.

[1] The proposition (19) was either repeated or obvious.

(24) (i) $(a-b)-e=(a-e)-b$,
(ii) $(a-b)-e=((-b)+a)-e$,
(iii) $(a-b)-e=((-b)-e)+a$,
(iv) $(a-b)-e=((-e)+a)-b$,
(v) $(a-b)-e=((-e)-b)+a$.

(25) $((-a)+b)-e=((-e)+b)-a$ and $((-a)+b)-e=((-e)-a)+b$.

(26) (i) $((-a)-b)-e=((-a)-e)-b$,
(ii) $((-a)-b)-e=((-b)-e)-a$,
(iii) $((-a)-b)-e=((-e)-a)-b$,
(iv) $((-a)-b)-e=((-e)-b)-a$.

(27) (i) $-((a+b)+e)=((-a)-b)-e$,
(ii) $-((a+b)-e)=((-a)-b)+e$,
(iii) $-((a-b)+e)=((-a)+b)-e$,
(iv) $-(((-a)+b)+e)=(a-b)-e$,
(v) $-((a-b)-e)=((-a)+b)+e$,
(vi) $-(((-a)+b)-e)=(a-b)+e$,
(vii) $-(((-a)-b)+e)=(a+b)-e$,
(viii) $-(((-a)-b)-e)=(a+b)+e$.

(28) (i) $a+e=(a+b)+(e-b)$,
(ii) $a+e=(b+a)+(e-b)$,
(iii) $a+e=(a-b)+(e+b)$,
(iv) $a+e=(a-b)+(b+e)$,
(v) $e+a=(a+b)+(e-b)$,
(vi) $e+a=(b+a)+(e-b)$,
(vii) $e+a=(a-b)+(e+b)$,
(viii) $e+a=(a-b)+(b+e)$,
(ix) $a+e=(a+b)-(b-e)$,
(x) $a+e=(b+a)-(b-e)$,
(xi) $e+a=(b+a)-(b-e)$,
(xii) $e+a=(a+b)-(b-e)$.

(29) (i) $a-e=(a-b)-(e-b)$,
(ii) $a-e=(a-b)+(b-e)$,
(iii) $a-e=(a+b)-(e+b)$,
(iv) $a-e=(b+a)-(e+b)$,
(v) $a-e=(b+a)-(b+e)$.

(30) If $b\neq 0$, then if $\frac{a}{b}=1$ or $a\cdot b^{-1}=1$ or $b^{-1}\cdot a=1$, then $a=b$.

(31) If $e\neq 0$ and $\frac{a}{e}=\frac{b}{e}$, then $a=b$.

Next we state a number of propositions:

(32) If $a\cdot 1=b\cdot 1$ or $a\cdot 1=1\cdot b$ or $1\cdot a=1\cdot b$ or $1\cdot a=b\cdot 1$, then $a=b$.

(33) If $a\neq 0$ and $b\neq 0$, then if $a^{-1}=b^{-1}$ or $\frac{1}{a}=\frac{1}{b}$ or $\frac{1}{a}=b^{-1}$, then $a=b$.

(34) If $b\neq 0$ and $\frac{a}{b}=-1$, then $a=-b$ and $b=-a$.

(35) If $a\cdot b=1$ or $b\cdot a=1$, then $a=\frac{1}{b}$ and $a=b^{-1}$.

(36) If $b \neq 0$, then if $a = \frac{1}{b}$ or $a = b^{-1}$, then $a \cdot b = 1$ and $b \cdot a = 1$ and $a^{-1} = b$ and $b = \frac{1}{a}$.

(37) If $b \neq 0$ but $a \cdot b = b$ or $b \cdot a = b$, then $a = 1$.

(38) If $b \neq 0$ but $a \cdot b = -b$ or $b \cdot a = -b$, then $a = -1$.

(39) If $a \neq 0$ and $b \neq 0$ and $\frac{b}{a} = b$, then $a = 1$.

(40) If $a \neq 0$ and $b \neq 0$ and $\frac{b}{a} = -b$, then $a = -1$.

(41) If $a \neq 0$, then $\frac{1}{a} \neq 0$.

(42) If $a \neq 0$ and $b \neq 0$, then $a \cdot b^{-1} \neq 0$ and $b^{-1} \cdot a \neq 0$ and $\frac{a}{b} \neq 0$ and $a^{-1} \cdot b^{-1} \neq 0$ and $\frac{1}{a \cdot b} \neq 0$.

(43) $\frac{1}{1} = 1$ and $1^{-1} = 1$ and $\frac{1}{-1} = -1$ and $(-1)^{-1} = -1$ and $(-1) \cdot (-1) = 1$.

(44) $\frac{a}{1} = a$ and $a \cdot 1^{-1} = a$ and $1^{-1} \cdot a = a$.

(45) If $a \neq 0$, then $\frac{-a}{a} = -1$ and $\frac{a}{-a} = -1$ and $(-a)^{-1} = -a^{-1}$.

(46) If $a \neq 0$, then if $a = a^{-1}$ or $a = \frac{1}{a}$, then $a = 1$ or $a = -1$.

(47) Suppose $a \neq 0$ and $b \neq 0$. Then
- (i) $(a \cdot b^{-1})^{-1} = a^{-1} \cdot b$,
- (ii) $(a \cdot b^{-1})^{-1} = b \cdot a^{-1}$,
- (iii) $(b^{-1} \cdot a)^{-1} = b \cdot a^{-1}$,
- (iv) $(b^{-1} \cdot a)^{-1} = a^{-1} \cdot b$,
- (v) $(a^{-1} \cdot b^{-1})^{-1} = a \cdot b$.

(48) If $a \neq 0$ and $b \neq 0$, then $\frac{1}{\frac{a}{b}} = \frac{b}{a}$ and $\frac{a}{b}^{-1} = \frac{b}{a}$.

(49) (i) $(-a) \cdot b = -b \cdot a$,
- (ii) $a \cdot (-b) = -b \cdot a$,
- (iii) $(-a) \cdot b = (-b) \cdot a$,
- (iv) $(-a) \cdot (-b) = a \cdot b$,
- (v) $(-a) \cdot (-b) = b \cdot a$,
- (vi) $-a \cdot (-b) = a \cdot b$,
- (vii) $-a \cdot (-b) = b \cdot a$,
- (viii) $-(-a) \cdot b = a \cdot b$,
- (ix) $-(-a) \cdot b = b \cdot a$.

(50) If $b \neq 0$, then $\frac{a}{b} = 0$ if and only if $a = 0$.

(51) If $a \neq 0$ and $b \neq 0$, then $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$.

(52) If $a \neq 0$, then $\frac{1}{\frac{1}{a}} = a$.

(53) Suppose $e \neq 0$ and $d \neq 0$. Then
- (i) $\frac{a}{e} \cdot \frac{b}{d} = \frac{b \cdot a}{e \cdot d}$,
- (ii) $\frac{a}{e} \cdot \frac{b}{d} = \frac{b \cdot a}{d \cdot e}$,
- (iii) $\frac{a}{e} \cdot \frac{b}{d} = \frac{a \cdot b}{d \cdot e}$,
- (iv) $\frac{a}{e} \cdot \frac{b}{d} = \frac{a}{d} \cdot \frac{b}{e}$.

(54) If $a \neq 0$, then $a \cdot \frac{1}{a} = 1$.

(55) Suppose $b \neq 0$ and $e \neq 0$. Then
- (i) $\frac{a}{b} = \frac{a \cdot e}{e \cdot b}$,

(ii) $\frac{a}{b} = \frac{e \cdot a}{b \cdot e}$,
(iii) $\frac{a}{b} = \frac{e \cdot a}{e \cdot b}$,
(iv) $\frac{a}{b} = \frac{\frac{a}{e}}{\frac{b}{e}}$,
(v) $\frac{a}{b} = e \cdot \frac{a}{b \cdot e}$,
(vi) $\frac{a}{b} = e \cdot \frac{a}{e \cdot b}$,
(vii) $\frac{a}{b} = \frac{a}{e \cdot b} \cdot e$,
(viii) $\frac{a}{b} = \frac{a}{b \cdot e} \cdot e$,
(ix) $\frac{a}{b} = e \cdot \frac{\frac{a}{e}}{b}$,
(x) $\frac{a}{b} = \frac{\frac{a}{e}}{b} \cdot e$,
(xi) $\frac{a}{b} = \frac{a}{e} \cdot \frac{e}{b}$.

(56) If $b \neq 0$, then $a \cdot \frac{1}{b} = \frac{a}{b}$ and $\frac{1}{b} \cdot a = \frac{a}{b}$.

(57) If $b \neq 0$, then $\frac{a}{\frac{1}{b}} = a \cdot b$ and $\frac{a}{\frac{1}{b}} = b \cdot a$.

(58) If $b \neq 0$, then $-\frac{a}{-b} = \frac{a}{b}$ and $-\frac{-a}{b} = \frac{a}{b}$ and $\frac{-a}{-b} = \frac{a}{b}$ and $\frac{-a}{b} = \frac{a}{-b}$.

(59) If $e \neq 0$, then $\frac{a+b}{e} = \frac{b}{e} + \frac{a}{e}$.

(60) Suppose $e \neq 0$ and $d \neq 0$. Then
(i) $\frac{a}{e} + \frac{b}{d} = \frac{d \cdot a + b \cdot e}{e \cdot d}$,
(ii) $\frac{a}{e} + \frac{b}{d} = \frac{d \cdot a + e \cdot b}{e \cdot d}$,
(iii) $\frac{a}{e} + \frac{b}{d} = \frac{a \cdot d + e \cdot b}{e \cdot d}$,
(iv) $\frac{a}{e} + \frac{b}{d} = \frac{d \cdot a + b \cdot e}{d \cdot e}$,
(v) $\frac{a}{e} + \frac{b}{d} = \frac{d \cdot a + e \cdot b}{d \cdot e}$,
(vi) $\frac{a}{e} + \frac{b}{d} = \frac{a \cdot d + e \cdot b}{d \cdot e}$,
(vii) $\frac{a}{e} - \frac{b}{d} = \frac{d \cdot a - b \cdot e}{e \cdot d}$,
(viii) $\frac{a}{e} - \frac{b}{d} = \frac{d \cdot a - e \cdot b}{e \cdot d}$,
(ix) $\frac{a}{e} - \frac{b}{d} = \frac{a \cdot d - e \cdot b}{e \cdot d}$,
(x) $\frac{a}{e} - \frac{b}{d} = \frac{d \cdot a - b \cdot e}{d \cdot e}$,
(xi) $\frac{a}{e} - \frac{b}{d} = \frac{d \cdot a - e \cdot b}{d \cdot e}$,
(xii) $\frac{a}{e} - \frac{b}{d} = \frac{a \cdot d - e \cdot b}{d \cdot e}$.

(61) Suppose $b \neq 0$ and $e \neq 0$. Then
(i) $\frac{\frac{a}{b}}{e} = \frac{e \cdot a}{b}$,
(ii) $\frac{\frac{a}{b}}{e} = a \cdot \frac{e}{b}$,
(iii) $\frac{\frac{a}{b}}{e} = \frac{e}{b} \cdot a$,
(iv) $\frac{\frac{a}{b}}{e} = e \cdot \frac{a}{b}$,
(v) $\frac{\frac{a}{b}}{e} = \frac{a}{b} \cdot e$.

(62) Suppose $b \neq 0$. Then
(i) $a = a \cdot \frac{b}{b}$,
(ii) $a = \frac{a \cdot b}{b}$,
(iii) $a = a \cdot (b \cdot \frac{1}{b})$,
(iv) $a = (a \cdot b) \cdot \frac{1}{b}$,

(v) $a = \frac{\frac{a}{b}}{b}$,

(vi) $a = \frac{a}{b} \cdot b$,

(vii) $a = \frac{a}{b \cdot \frac{1}{b}}$,

(viii) $a = a \cdot (\frac{1}{b} \cdot b)$,

(ix) $a = (a \cdot \frac{1}{b}) \cdot b$,

(x) $a = b \cdot \frac{a}{b}$,

(xi) $a = \frac{b \cdot a}{b}$,

(xii) $a = b \cdot (a \cdot \frac{1}{b})$,

(xiii) $a = (b \cdot a) \cdot \frac{1}{b}$,

(xiv) $a = \frac{b}{b} \cdot a$,

(xv) $a = (\frac{1}{b} \cdot b) \cdot a$,

(xvi) $a = \frac{1}{b} \cdot (b \cdot a)$,

(xvii) $a = \frac{1}{b} \cdot (a \cdot b)$,

(xviii) $a = (\frac{1}{b} \cdot a) \cdot b$.

The following propositions are true:

(63) For every a, b there exists e such that $a = b - e$.

(64) For all a, b such that $a \neq 0$ and $b \neq 0$ there exists e such that $a = \frac{b}{e}$.

(65) Suppose $b \neq 0$. Then

(i) $\frac{a}{b} + e = \frac{a + e \cdot b}{b}$,

(ii) $\frac{a}{b} + e = \frac{a + b \cdot e}{b}$,

(iii) $\frac{a}{b} + e = \frac{b \cdot e + a}{b}$,

(iv) $\frac{a}{b} + e = \frac{e \cdot b + a}{b}$,

(v) $e + \frac{a}{b} = \frac{e \cdot b + a}{b}$,

(vi) $e + \frac{a}{b} = \frac{a + e \cdot b}{b}$,

(vii) $e + \frac{a}{b} = \frac{a + b \cdot e}{b}$,

(viii) $e + \frac{a}{b} = \frac{b \cdot e + a}{b}$.

(66) Suppose $b \neq 0$. Then

(i) $\frac{a}{b} - e = \frac{a - e \cdot b}{b}$,

(ii) $\frac{a}{b} - e = \frac{a - b \cdot e}{b}$,

(iii) $e - \frac{a}{b} = \frac{e \cdot b - a}{b}$,

(iv) $e - \frac{a}{b} = \frac{b \cdot e - a}{b}$.

(67) Suppose $b \neq 0$ and $e \neq 0$. Then

(i) $\frac{\frac{a}{b}}{e} = \frac{a}{b \cdot e}$,

(ii) $\frac{\frac{a}{b}}{e} = \frac{a}{e \cdot b}$,

(iii) $\frac{\frac{a}{b}}{e} = \frac{\frac{a}{e}}{b}$,

(iv) $\frac{\frac{a}{b}}{e} = \frac{1}{b} \cdot \frac{a}{e}$,

(v) $\frac{\frac{a}{b}}{e} = \frac{a}{e} \cdot \frac{1}{b}$,

(vi) $\frac{\frac{a}{b}}{e} = \frac{1}{e} \cdot \frac{a}{b}$,

(vii) $\frac{\frac{a}{b}}{e} = \frac{a}{b} \cdot \frac{1}{e}$,

(viii) $\frac{1}{e} \cdot \frac{a}{b} = \frac{a}{b \cdot e}$,
(ix) $\frac{1}{e} \cdot \frac{a}{b} = \frac{a}{e \cdot b}$,
(x) $\frac{a}{b} \cdot \frac{1}{e} = \frac{a}{e \cdot b}$,
(xi) $\frac{a}{b} \cdot \frac{1}{e} = \frac{a}{b \cdot e}$.

(68) Suppose $b \neq 0$. Then $e \cdot \frac{a}{b} = \frac{e \cdot a}{b}$ and $e \cdot \frac{a}{b} = \frac{a \cdot e}{b}$ and $\frac{a}{b} \cdot e = \frac{a \cdot e}{b}$ and $\frac{a}{b} \cdot e = \frac{e \cdot a}{b}$.

(69) $(a \cdot b) \cdot e = (a \cdot e) \cdot b$ and $(a \cdot b) \cdot e = (b \cdot e) \cdot a$ and $(a \cdot b) \cdot e = (e \cdot a) \cdot b$ and $(a \cdot b) \cdot e = (e \cdot b) \cdot a$.

(70) Suppose $e \neq 0$ and $d \neq 0$. Then
(i) $\frac{a \cdot b}{e \cdot d} = \frac{\frac{a}{e} \cdot b}{d}$,
(ii) $\frac{a \cdot b}{e \cdot d} = \frac{b \cdot \frac{a}{e}}{d}$,
(iii) $\frac{a \cdot b}{e \cdot d} = \frac{\frac{b}{e} \cdot a}{d}$,
(iv) $\frac{a \cdot b}{e \cdot d} = \frac{a \cdot \frac{b}{e}}{d}$,
(v) $\frac{a \cdot b}{d \cdot e} = \frac{\frac{a}{e} \cdot b}{d}$,
(vi) $\frac{a \cdot b}{d \cdot e} = \frac{b \cdot \frac{a}{e}}{d}$,
(vii) $\frac{a \cdot b}{d \cdot e} = \frac{a \cdot \frac{b}{e}}{d}$,
(viii) $\frac{a \cdot b}{d \cdot e} = \frac{\frac{b}{e} \cdot a}{d}$.

(71) $(-1) \cdot a = -a$ and $a \cdot (-1) = -a$ and $(-a) \cdot (-1) = a$ and $(-1) \cdot (-a) = a$ and $-a = \frac{a}{-1}$ and $a = \frac{-a}{-1}$.

(72) If $e \neq 0$, then if $a \cdot e = b$ or $e \cdot a = b$, then $a = \frac{b}{e}$.

(73) If $e \neq 0$ and $a = \frac{b}{e}$, then $a \cdot e = b$ and $e \cdot a = b$.

(74) If $a \neq 0$ and $e \neq 0$ and $a = \frac{b}{e}$, then $e = \frac{b}{a}$.

(75) If $e \neq 0$ and $d \neq 0$, then if $a \cdot e = b \cdot d$ or $e \cdot a = b \cdot d$ or $e \cdot a = d \cdot b$ or $a \cdot e = d \cdot b$, then $\frac{a}{d} = \frac{b}{e}$.

(76) If $e \neq 0$ and $d \neq 0$ and $\frac{a}{d} = \frac{b}{e}$, then $a \cdot e = b \cdot d$ and $e \cdot a = b \cdot d$ and $e \cdot a = d \cdot b$ and $a \cdot e = d \cdot b$.

(77) If $e \neq 0$ and $d \neq 0$, then if $a \cdot e = \frac{b}{d}$ or $e \cdot a = \frac{b}{d}$, then $a \cdot d = \frac{b}{e}$ and $d \cdot a = \frac{b}{e}$.

(78) Suppose $b \neq 0$. Then
(i) $a \cdot e = (a \cdot b) \cdot \frac{e}{b}$,
(ii) $a \cdot e = (b \cdot a) \cdot \frac{e}{b}$,
(iii) $a \cdot e = \frac{a}{b} \cdot (e \cdot b)$,
(iv) $a \cdot e = \frac{a}{b} \cdot (b \cdot e)$,
(v) $e \cdot a = (a \cdot b) \cdot \frac{e}{b}$,
(vi) $e \cdot a = (b \cdot a) \cdot \frac{e}{b}$,
(vii) $e \cdot a = \frac{a}{b} \cdot (e \cdot b)$,
(viii) $e \cdot a = \frac{a}{b} \cdot (b \cdot e)$.

(79) Suppose $b \neq 0$ and $e \neq 0$. Then $a \cdot e = \frac{a \cdot b}{\frac{b}{e}}$ and $a \cdot e = \frac{b \cdot a}{\frac{b}{e}}$ and $e \cdot a = \frac{b \cdot a}{\frac{b}{e}}$ and $e \cdot a = \frac{a \cdot b}{\frac{b}{e}}$.

(80) If $b \neq 0$, then $\frac{a}{b} \cdot e = \frac{e}{b} \cdot a$ and $\frac{a}{b} \cdot e = (\frac{1}{b} \cdot a) \cdot e$ and $\frac{a}{b} \cdot e = (\frac{1}{b} \cdot e) \cdot a$.

(81) $(-a) \cdot (-b) = a \cdot b$ and $(-a) \cdot (-b) = b \cdot a$.

(82) If $b \neq 0$ and $d \neq 0$ and $b \neq d$ and $\frac{a}{b} = \frac{e}{d}$, then $\frac{a}{b} = \frac{a-e}{b-d}$.

(83) Suppose $b \neq 0$ and $d \neq 0$ and $b \neq -d$ and $\frac{a}{b} = \frac{e}{d}$. Then $\frac{a}{b} = \frac{a+e}{b+d}$ and $\frac{a}{b} = \frac{e+a}{b+d}$ and $\frac{a}{b} = \frac{e+a}{d+b}$ and $\frac{a}{b} = \frac{a+e}{d+b}$.

(84) (i) $e \cdot (a+b) = a \cdot e + e \cdot b$,
(ii) $e \cdot (a+b) = e \cdot a + b \cdot e$,
(iii) $e \cdot (a+b) = a \cdot e + b \cdot e$,
(iv) $(a+b) \cdot e = e \cdot a + b \cdot e$,
(v) $(a+b) \cdot e = a \cdot e + e \cdot b$,
(vi) $(a+b) \cdot e = e \cdot a + e \cdot b$,
(vii) $e \cdot (b+a) = a \cdot e + e \cdot b$,
(viii) $e \cdot (b+a) = e \cdot a + b \cdot e$,
(ix) $e \cdot (b+a) = a \cdot e + b \cdot e$,
(x) $(b+a) \cdot e = e \cdot a + b \cdot e$,
(xi) $(b+a) \cdot e = a \cdot e + e \cdot b$,
(xii) $(b+a) \cdot e = e \cdot a + e \cdot b$,
(xiii) $(a+b) \cdot e = b \cdot e + a \cdot e$,
(xiv) $e \cdot (a+b) = e \cdot b + e \cdot a$.

(85) (i) $e \cdot (a-b) = a \cdot e - e \cdot b$,
(ii) $e \cdot (a-b) = e \cdot a - b \cdot e$,
(iii) $e \cdot (a-b) = a \cdot e - b \cdot e$,
(iv) $(a-b) \cdot e = e \cdot a - b \cdot e$,
(v) $(a-b) \cdot e = a \cdot e - e \cdot b$,
(vi) $(a-b) \cdot e = e \cdot a - e \cdot b$,
(vii) $(a-b) \cdot e = (b-a) \cdot (-e)$,
(viii) $(a-b) \cdot e = -(b-a) \cdot e$,
(ix) $e \cdot (a-b) = (-e) \cdot (b-a)$,
(x) $e \cdot (a-b) = -e \cdot (b-a)$.

(86) If $a \neq 0$, then if $\frac{1}{a} = 1$ or $a^{-1} = 1$, then $a = 1$.

(87) If $a \neq 0$, then if $\frac{1}{a} = -1$ or $a^{-1} = -1$, then $a = -1$.

(88) (i) $2 \cdot a = a + a$,
(ii) $a \cdot 2 = a + a$,
(iii) $3 \cdot a = (a+a) + a$,
(iv) $a \cdot 3 = (a+a) + a$,
(v) $4 \cdot a = ((a+a)+a) + a$,
(vi) $a \cdot 4 = ((a+a)+a) + a$.

(89) $\frac{a+a}{2} = a$ and $\frac{(a+a)+a}{3} = a$ and $\frac{((a+a)+a)+a}{4} = a$ and $\frac{a+a}{4} = \frac{a}{2}$.

(90) (i) $\frac{a}{2} + \frac{a}{2} = a$,
(ii) $(\frac{a}{3} + \frac{a}{3}) + \frac{a}{3} = a$,
(iii) $((\frac{a}{4} + \frac{a}{4}) + \frac{a}{4}) + \frac{a}{4} = a$,
(iv) $\frac{a}{4} + \frac{a}{4} = \frac{a}{2}$.

(91) If $b \neq 0$, then $\frac{a}{2 \cdot b} + \frac{a}{2 \cdot b} = \frac{a}{b}$ and $(\frac{a}{3 \cdot b} + \frac{a}{3 \cdot b}) + \frac{a}{3 \cdot b} = \frac{a}{b}$.

(92) Suppose $e \neq 0$. Then
(i) $a + b = e \cdot (\frac{a}{e} + \frac{b}{e})$,
(ii) $b + a = e \cdot (\frac{a}{e} + \frac{b}{e})$,
(iii) $b + a = (\frac{a}{e} + \frac{b}{e}) \cdot e$,
(iv) $a + b = (\frac{a}{e} + \frac{b}{e}) \cdot e$.

(93) If $e \neq 0$, then $a - b = e \cdot (\frac{a}{e} - \frac{b}{e})$ and $a - b = (\frac{a}{e} - \frac{b}{e}) \cdot e$.

One can prove the following propositions:

(94) Suppose $e \neq 0$. Then
(i) $a + b = \frac{a \cdot e + b \cdot e}{e}$,
(ii) $a + b = \frac{a \cdot e + e \cdot b}{e}$,
(iii) $a + b = \frac{e \cdot a + e \cdot b}{e}$,
(iv) $a + b = \frac{e \cdot a + b \cdot e}{e}$,
(v) $b + a = \frac{e \cdot a + b \cdot e}{e}$,
(vi) $b + a = \frac{e \cdot a + e \cdot b}{e}$,
(vii) $b + a = \frac{a \cdot e + e \cdot b}{e}$,
(viii) $b + a = \frac{a \cdot e + b \cdot e}{e}$.

(95) Suppose $e \neq 0$. Then
(i) $a - b = \frac{a \cdot e - b \cdot e}{e}$,
(ii) $a - b = \frac{a \cdot e - e \cdot b}{e}$,
(iii) $a - b = \frac{e \cdot a - e \cdot b}{e}$,
(iv) $a - b = \frac{e \cdot a - b \cdot e}{e}$.

(96) Suppose $a \neq 0$. Then
(i) $a + b = a \cdot (1 + \frac{b}{a})$,
(ii) $a + b = (1 + \frac{b}{a}) \cdot a$,
(iii) $a + b = (\frac{b}{a} + 1) \cdot a$,
(iv) $a + b = a \cdot (\frac{b}{a} + 1)$,
(v) $b + a = a \cdot (1 + \frac{b}{a})$,
(vi) $b + a = (1 + \frac{b}{a}) \cdot a$,
(vii) $b + a = (\frac{b}{a} + 1) \cdot a$,
(viii) $b + a = a \cdot (\frac{b}{a} + 1)$.

(97) If $a \neq 0$, then $a - b = a \cdot (1 - \frac{b}{a})$ and $a - b = (1 - \frac{b}{a}) \cdot a$.

(98) $(a - b) \cdot (e - d) = (b - a) \cdot (d - e)$.

(99) (i) $((a + b) + e) \cdot d = (a \cdot d + b \cdot d) + e \cdot d$,
(ii) $d \cdot ((a + b) + e) = (d \cdot a + d \cdot b) + d \cdot e$,
(iii) $((a + b) - e) \cdot d = (a \cdot d + b \cdot d) - e \cdot d$,

(iv) $d \cdot ((a+b)-e) = (d \cdot a + d \cdot b) - d \cdot e$,
(v) $((a-b)+e) \cdot d = (a \cdot d - b \cdot d) + e \cdot d$,
(vi) $d \cdot ((a-b)+e) = (d \cdot a - d \cdot b) + d \cdot e$,
(vii) $((a-b)-e) \cdot d = (a \cdot d - b \cdot d) - e \cdot d$,
(viii) $d \cdot ((a-b)-e) = (d \cdot a - d \cdot b) - d \cdot e$.

(100) Suppose $d \neq 0$. Then
(i) $\frac{(a+b)+e}{d} = (\frac{a}{d} + \frac{b}{d}) + \frac{e}{d}$,
(ii) $\frac{(a+b)-e}{d} = (\frac{a}{d} + \frac{b}{d}) - \frac{e}{d}$,
(iii) $\frac{(a-b)+e}{d} = (\frac{a}{d} - \frac{b}{d}) + \frac{e}{d}$,
(iv) $\frac{(a-b)-e}{d} = (\frac{a}{d} - \frac{b}{d}) - \frac{e}{d}$.

(101) (i) $(a+b) \cdot (e+d) = ((a \cdot e + a \cdot d) + b \cdot e) + b \cdot d$,
(ii) $(a+b) \cdot (e-d) = ((a \cdot e - a \cdot d) + b \cdot e) - b \cdot d$,
(iii) $(a-b) \cdot (e+d) = ((a \cdot e + a \cdot d) - b \cdot e) - b \cdot d$,
(iv) $(a-b) \cdot (e-d) = ((a \cdot e - a \cdot d) - b \cdot e) + b \cdot d$.

(103)[2] If $a \geq b$, then $a+e \geq e+b$ and $e+a \geq e+b$ and $e+a \geq b+e$.

(104) If $a+e \geq b+e$ or $a+e \geq e+b$ or $e+a \geq e+b$ or $e+a \geq b+e$ or $a-e \geq b-e$, then $a \geq b$.

(105) Suppose that
(i) $a-b \leq 0$ or $a+(-b) \leq 0$ or $(-b)+a \leq 0$ or $-a \geq -b$ or $b-a \geq 0$ or $b+(-a) \geq 0$ or $(-a)+b \geq 0$ or $a-e \leq b+(-e)$ or $a-e \leq (-e)+b$ or $a+(-e) \leq b-e$ or $(-e)+a \leq b-e$ or $e-a \geq e-b$.
Then $a \leq b$.

(106) Suppose that
(i) $a-b < 0$ or $a+(-b) < 0$ or $(-b)+a < 0$ or $-a > -b$ or $b-a > 0$ or $b+(-a) > 0$ or $(-a)+b > 0$ or $a-e < b+(-e)$ or $a-e < (-e)+b$ or $a+(-e) < b-e$ or $(-e)+a < b-e$ or $e-a > e-b$.
Then $a < b$.

(107) Suppose $a \leq b$. Then $a-b \leq 0$ and $a+(-b) \leq 0$ and $(-b)+a \leq 0$ and $b-a \geq 0$ and $b+(-a) \geq 0$ and $(-a)+b \geq 0$ and $-a \geq -b$ and $e-a \geq e-b$.

(108) Suppose $a < b$. Then $a-b < 0$ and $a+(-b) < 0$ and $(-b)+a < 0$ and $b-a > 0$ and $b+(-a) > 0$ and $(-a)+b > 0$ and $-a > -b$ and $e-a > e-b$.

(109) If $a \leq -b$, then $a+b \leq 0$ and $b+a \leq 0$ and $-a \geq b$.

(110) If $a < -b$, then $a+b < 0$ and $b+a < 0$ and $-a > b$.

(111) If $-a \leq b$, then $b+a \geq 0$ and $a+b \geq 0$ and $a \geq -b$.

(112) If $-b < a$, then $a+b > 0$ and $b+a > 0$ and $b > -a$.

(113) If $a+b \leq 0$ or $b+a \leq 0$, then $a \leq -b$.

(114) If $a+b < 0$ or $b+a < 0$, then $a < -b$.

(115) If $a+b \geq 0$ or $b+a \geq 0$, then $a \geq -b$.

[2] The proposition (102) was either repeated or obvious.

(116) If $a + b > 0$ or $b + a > 0$, then $a > -b$.

(117) Suppose $b > 0$. Then
(i) if $\frac{a}{b} > 1$, then $a > b$,
(ii) if $\frac{a}{b} < 1$, then $a < b$,
(iii) if $\frac{a}{b} > -1$, then $a > -b$ and $b > -a$,
(iv) if $\frac{a}{b} < -1$, then $a < -b$ and $b < -a$.

(118) Suppose $b > 0$. Then
(i) if $\frac{a}{b} \geq 1$, then $a \geq b$,
(ii) if $\frac{a}{b} \leq 1$, then $a \leq b$,
(iii) if $\frac{a}{b} \geq -1$, then $a \geq -b$ and $b \geq -a$,
(iv) if $\frac{a}{b} \leq -1$, then $a \leq -b$ and $b \leq -a$.

(119) Suppose $b < 0$. Then
(i) if $\frac{a}{b} > 1$, then $a < b$,
(ii) if $\frac{a}{b} < 1$, then $a > b$,
(iii) if $\frac{a}{b} > -1$, then $a < -b$ and $b < -a$,
(iv) if $\frac{a}{b} < -1$, then $a > -b$ and $b > -a$.

(120) Suppose $b < 0$. Then
(i) if $\frac{a}{b} \geq 1$, then $a \leq b$,
(ii) if $\frac{a}{b} \leq 1$, then $a \geq b$,
(iii) if $\frac{a}{b} \geq -1$, then $a \leq -b$ and $b \leq -a$,
(iv) if $\frac{a}{b} \leq -1$, then $a \geq -b$ and $b \geq -a$.

(121) If $a \geq 0$ or $a > 0$ but $b \geq 0$ or $b > 0$ or $a \leq 0$ or $a < 0$ but $b \leq 0$ or $b < 0$, then $a \cdot b \geq 0$ and $b \cdot a \geq 0$.

(122) If $a < 0$ and $b < 0$ or $a > 0$ and $b > 0$, then $a \cdot b > 0$.

(123) If $a \geq 0$ or $a > 0$ but $b \leq 0$ or $b < 0$ or $a \leq 0$ or $a < 0$ but $b \geq 0$ or $b > 0$, then $a \cdot b \leq 0$ and $b \cdot a \leq 0$.

(124) If $a > 0$ and $b < 0$, then $a \cdot b < 0$ and $b \cdot a < 0$.

One can prove the following propositions:

(125) If $a \leq 0$ and $b < 0$ or $a \geq 0$ and $b > 0$, then $\frac{a}{b} \geq 0$.

(126) If $a \geq 0$ and $b < 0$ or $a \leq 0$ and $b > 0$, then $\frac{a}{b} \leq 0$.

(127) If $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$, then $\frac{a}{b} > 0$.

(128) If $a < 0$ and $b > 0$, then $\frac{a}{b} < 0$ and $\frac{b}{a} < 0$.

(129) If $a \cdot b \leq 0$, then $a \geq 0$ and $b \leq 0$ or $a \leq 0$ and $b \geq 0$.

(131)[3] If $a \cdot b > 0$, then $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.

(132) If $a \cdot b < 0$, then $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.

(133) If $b \neq 0$ and $\frac{a}{b} \leq 0$, then $b > 0$ and $a \leq 0$ or $b < 0$ and $a \geq 0$.

(134) If $b \neq 0$ and $\frac{a}{b} \geq 0$, then $b > 0$ and $a \geq 0$ or $b < 0$ and $a \leq 0$.

(135) If $b \neq 0$ and $\frac{a}{b} < 0$, then $b < 0$ and $a > 0$ or $b > 0$ and $a < 0$.

(136) If $b \neq 0$ and $\frac{a}{b} > 0$, then $b > 0$ and $a > 0$ or $b < 0$ and $a < 0$.

[3]The proposition (130) was either repeated or obvious.

(137) If $a > 1$ but $b > 1$ or $b \geq 1$ or $a < -1$ but $b < -1$ or $b \leq -1$, then $a \cdot b > 1$ and $b \cdot a > 1$.

(138) If $a \geq 1$ and $b \geq 1$ or $a \leq -1$ and $b \leq -1$, then $a \cdot b \geq 1$.

(139) Suppose that

(i) $0 < a$ or $0 \leq a$ but $a < 1$ but $0 < b$ or $0 \leq b$ but $b < 1$ or $b \leq 1$ or $0 > a$ or $0 \geq a$ but $a > -1$ but $0 > b$ or $0 \geq b$ but $b > -1$ or $b \geq -1$.

Then $a \cdot b < 1$ and $b \cdot a < 1$.

(140) If $0 \leq a$ and $a \leq 1$ and $0 \leq b$ and $b \leq 1$ or $0 \geq a$ and $a \geq -1$ and $0 \geq b$ and $b \geq -1$, then $a \cdot b \leq 1$.

(141) If $e < 0$ and $a \leq b$ or $e > 0$ and $a \geq b$, then $\frac{a}{e} \geq \frac{b}{e}$.

(142) If $0 < a$ and $a < b$ or $b < a$ and $a < 0$, then $\frac{a}{b} < 1$ and $\frac{b}{a} > 1$.

(143) If $0 < a$ and $a \leq b$ or $b \leq a$ and $a < 0$, then $\frac{a}{b} \leq 1$ and $\frac{b}{a} \geq 1$.

(144) If $a > 0$ and $b > 1$ or $a < 0$ and $b < 1$, then $a \cdot b > a$ and $b \cdot a > a$.

(145) If $a > 0$ and $b < 1$ or $a < 0$ and $b > 1$, then $a \cdot b < a$ and $b \cdot a < a$.

(146) If $a > 0$ or $a \geq 0$ but $b > 1$ or $b \geq 1$ or $a < 0$ or $a \leq 0$ but $b < 1$ or $b \leq 1$, then $a \cdot b \geq a$ and $b \cdot a \geq a$.

(147) If $a > 0$ or $a \geq 0$ but $b < 1$ or $b \leq 1$ or $a < 0$ or $a \leq 0$ but $b > 1$ or $b \geq 1$, then $a \cdot b \leq a$ and $b \cdot a \leq a$.

(148) $a > 0$ if and only if $-a < 0$ but $a \geq 0$ if and only if $-a \leq 0$ but $a \leq 0$ if and only if $-a \geq 0$.

(149) If $a < 0$, then $\frac{1}{a} < 0$ and $a^{-1} < 0$ but if $a > 0$, then $\frac{1}{a} > 0$.

(150) If $a \neq 0$, then if $\frac{1}{a} < 0$, then $a < 0$ but if $\frac{1}{a} > 0$, then $a > 0$.

(151) If $0 < a$ or $b < 0$ but $a < b$, then $\frac{1}{a} > \frac{1}{b}$.

(152) If $0 < a$ or $b < 0$ but $a \leq b$, then $\frac{1}{a} \geq \frac{1}{b}$.

(153) If $a < 0$ and $b > 0$, then $\frac{1}{a} < \frac{1}{b}$.

(154) If $a \neq 0$ and $b \neq 0$ but $\frac{1}{b} > 0$ or $\frac{1}{a} < 0$ and $\frac{1}{a} > \frac{1}{b}$, then $a < b$.

(155) If $a \neq 0$ and $b \neq 0$ but $\frac{1}{b} > 0$ or $\frac{1}{a} < 0$ and $\frac{1}{a} \geq \frac{1}{b}$, then $a \leq b$.

Next we state a number of propositions:

(156) If $a \neq 0$ and $b \neq 0$ and $\frac{1}{a} < 0$ and $\frac{1}{b} > 0$, then $a < b$.

(157) If $a < -1$, then $0 > \frac{1}{a}$ and $\frac{1}{a} > -1$.

(158) If $a \leq -1$, then $0 > \frac{1}{a}$ and $\frac{1}{a} \geq -1$.

(159) If $-1 < a$ and $a < 0$, then $\frac{1}{a} < -1$.

(160) If $-1 \leq a$ and $a < 0$, then $\frac{1}{a} \leq -1$.

(161) If $0 < a$ and $a < 1$, then $\frac{1}{a} > 1$.

(162) If $0 < a$ and $a \leq 1$, then $\frac{1}{a} \geq 1$.

(163) If $1 < a$, then $0 < \frac{1}{a}$ and $\frac{1}{a} < 1$.

(164) If $1 \leq a$, then $0 < \frac{1}{a}$ and $\frac{1}{a} \leq 1$.

(165) If $b \leq e - a$, then $a \leq e - b$ but if $b \geq e - a$, then $a \geq e - b$.

(166) If $b < e - a$, then $a < e - b$ but if $b > e - a$, then $a > e - b$.

(167) If $a + b \leq e + d$, then $a - e \leq d - b$ and $e - a \geq b - d$ and $a - d \leq e - b$ and $d - a \geq b - e$.

(168) If $a + b < e + d$, then $a - e < d - b$ and $e - a > b - d$ and $a - d < e - b$ and $d - a > b - e$.

(169) Suppose $a - b \leq e - d$. Then $a + d \leq e + b$ and $d + a \leq e + b$ and $d + a \leq b + e$ and $a + d \leq b + e$ and $a - e \leq b - d$ and $e - a \geq d - b$ and $b - a \geq d - e$.

(170) Suppose $a - b < e - d$. Then $a + d < e + b$ and $d + a < e + b$ and $d + a < b + e$ and $a + d < b + e$ and $a - e < b - d$ and $e - a > d - b$ and $b - a > d - e$.

(171) (i) If $a + b \leq e - d$ or $b + a \leq e - d$, then $a + d \leq e - b$ and $d + a \leq e - b$,
(ii) if $a + b \geq e - d$ or $b + a \geq e - d$, then $a + d \geq e - b$ and $d + a \geq e - b$.

(172) (i) If $a + b < e - d$ or $b + a < e - d$, then $a + d < e - b$ and $d + a < e - b$,
(ii) if $a + b > e - d$ or $b + a > e - d$, then $a + d > e - b$ and $d + a > e - b$.

(173) If $a < 0$, then $b + a < b$ and $a + b < b$ and $b - a > b$ but if $a + b < b$ or $b + a < b$ or $b - a > b$, then $a < 0$.

(174) If $a \leq 0$, then $b + a \leq b$ and $a + b \leq b$ and $b - a \geq b$ but if $b + a \leq b$ or $a + b \leq b$ or $b - a \geq b$, then $a \leq 0$.

(175) If $a > 0$, then $b + a > b$ and $a + b > b$ and $b - a < b$ but if $b + a > b$ or $a + b > b$ or $b - a < b$, then $a > 0$.

(176) If $a \geq 0$, then $b + a \geq b$ and $a + b \geq b$ and $b - a \leq b$ but if $b + a \geq b$ or $a + b \geq b$ or $b - a \leq b$, then $a \geq 0$.

(177) (i) If $b > 0$ but $a \cdot b \leq e$ or $b \cdot a \leq e$, then $a \leq \frac{e}{b}$,
(ii) if $b < 0$ but $a \cdot b \leq e$ or $b \cdot a \leq e$, then $a \geq \frac{e}{b}$,
(iii) if $b > 0$ but $a \cdot b \geq e$ or $b \cdot a \geq e$, then $a \geq \frac{e}{b}$,
(iv) if $b < 0$ but $a \cdot b \geq e$ or $b \cdot a \geq e$, then $a \leq \frac{e}{b}$.

(178) (i) If $b > 0$ but $a \cdot b < e$ or $b \cdot a < e$, then $a < \frac{e}{b}$,
(ii) if $b < 0$ but $a \cdot b < e$ or $b \cdot a < e$, then $a > \frac{e}{b}$,
(iii) if $b > 0$ but $a \cdot b > e$ or $b \cdot a > e$, then $a > \frac{e}{b}$,
(iv) if $b < 0$ but $a \cdot b > e$ or $b \cdot a > e$, then $a < \frac{e}{b}$.

(179) (i) If $b > 0$ and $a \geq \frac{e}{b}$, then $a \cdot b \geq e$ and $b \cdot a \geq e$,
(ii) if $b > 0$ and $a \leq \frac{e}{b}$, then $a \cdot b \leq e$ and $b \cdot a \leq e$,
(iii) if $b < 0$ and $a \geq \frac{e}{b}$, then $a \cdot b \leq e$ and $b \cdot a \leq e$,
(iv) if $b < 0$ and $a \leq \frac{e}{b}$, then $a \cdot b \geq e$ and $b \cdot a \geq e$.

(180) (i) If $b > 0$ and $a > \frac{e}{b}$, then $a \cdot b > e$ and $b \cdot a > e$,
(ii) if $b > 0$ and $a < \frac{e}{b}$, then $a \cdot b < e$ and $b \cdot a < e$,
(iii) if $b < 0$ and $a > \frac{e}{b}$, then $a \cdot b < e$ and $b \cdot a < e$,
(iv) if $b < 0$ and $a < \frac{e}{b}$, then $a \cdot b > e$ and $b \cdot a > e$.

(181) If for every a such that $a > 0$ holds $b + a \geq e$ or for every a such that $a < 0$ holds $b - a \geq e$, then $b \geq e$.

(182) If for every a such that $a > 0$ holds $b - a \leq e$ or for every a such that $a < 0$ holds $b + a \leq e$, then $b \leq e$.

(183) If for every a such that $a > 1$ holds $b \cdot a \geq e$ or for every a such that $0 < a$ and $a < 1$ holds $\frac{b}{a} \geq e$, then $b \geq e$.

(184) If for every a such that $0 < a$ and $a < 1$ holds $b \cdot a \leq e$ or for every a such that $a > 1$ holds $\frac{b}{a} \leq e$, then $b \leq e$.

(185) Suppose $b > 0$ and $d > 0$ or $b < 0$ and $d < 0$ but $a \cdot d < e \cdot b$ or $d \cdot a < e \cdot b$ or $d \cdot a < b \cdot e$ or $a \cdot d < b \cdot e$. Then $\frac{a}{b} < \frac{e}{d}$.

(186) Suppose $b > 0$ and $d < 0$ or $b < 0$ and $d > 0$ but $a \cdot d < e \cdot b$ or $d \cdot a < e \cdot b$ or $d \cdot a < b \cdot e$ or $a \cdot d < b \cdot e$. Then $\frac{a}{b} > \frac{e}{d}$.

The following propositions are true:

(187) Suppose $b > 0$ and $d > 0$ or $b < 0$ and $d < 0$ but $a \cdot d \leq e \cdot b$ or $d \cdot a \leq e \cdot b$ or $d \cdot a \leq b \cdot e$ or $a \cdot d \leq b \cdot e$. Then $\frac{a}{b} \leq \frac{e}{d}$.

(188) Suppose $b > 0$ and $d < 0$ or $b < 0$ and $d > 0$ but $a \cdot d \leq e \cdot b$ or $d \cdot a \leq e \cdot b$ or $d \cdot a \leq b \cdot e$ or $a \cdot d \leq b \cdot e$. Then $\frac{a}{b} \geq \frac{e}{d}$.

(189) Suppose $b > 0$ and $d > 0$ or $b < 0$ and $d < 0$ but $\frac{a}{b} < \frac{e}{d}$. Then $a \cdot d < e \cdot b$ and $d \cdot a < e \cdot b$ and $d \cdot a < b \cdot e$ and $a \cdot d < b \cdot e$.

(190) Suppose $b < 0$ and $d > 0$ or $b > 0$ and $d < 0$ but $\frac{a}{b} < \frac{e}{d}$. Then $a \cdot d > e \cdot b$ and $d \cdot a > e \cdot b$ and $d \cdot a > b \cdot e$ and $a \cdot d > b \cdot e$.

(191) Suppose $b > 0$ and $d > 0$ or $b < 0$ and $d < 0$ but $\frac{a}{b} \leq \frac{e}{d}$. Then $a \cdot d \leq e \cdot b$ and $d \cdot a \leq e \cdot b$ and $d \cdot a \leq b \cdot e$ and $a \cdot d \leq b \cdot e$.

(192) Suppose $b < 0$ and $d > 0$ or $b > 0$ and $d < 0$ but $\frac{a}{b} \leq \frac{e}{d}$. Then $a \cdot d \geq e \cdot b$ and $d \cdot a \geq e \cdot b$ and $d \cdot a \geq b \cdot e$ and $a \cdot d \geq b \cdot e$.

(193) Suppose $b < 0$ and $d < 0$ or $b > 0$ and $d > 0$. Then
(i) if $a \cdot b < \frac{e}{d}$ or $b \cdot a < \frac{e}{d}$, then $a \cdot d < \frac{e}{b}$ and $d \cdot a < \frac{e}{b}$,
(ii) if $a \cdot b > \frac{e}{d}$ or $b \cdot a > \frac{e}{d}$, then $a \cdot d > \frac{e}{b}$ and $d \cdot a > \frac{e}{b}$.

(194) Suppose $b < 0$ and $d > 0$ or $b > 0$ and $d < 0$. Then
(i) if $a \cdot b < \frac{e}{d}$ or $b \cdot a < \frac{e}{d}$, then $a \cdot d > \frac{e}{b}$ and $d \cdot a > \frac{e}{b}$,
(ii) if $a \cdot b > \frac{e}{d}$ or $b \cdot a > \frac{e}{d}$, then $a \cdot d < \frac{e}{b}$ and $d \cdot a < \frac{e}{b}$.

(195) Suppose $b < 0$ and $d < 0$ or $b > 0$ and $d > 0$. Then
(i) if $a \cdot b \leq \frac{e}{d}$ or $b \cdot a \leq \frac{e}{d}$, then $a \cdot d \leq \frac{e}{b}$ and $d \cdot a \leq \frac{e}{b}$,
(ii) if $a \cdot b \geq \frac{e}{d}$ or $b \cdot a \geq \frac{e}{d}$, then $a \cdot d \geq \frac{e}{b}$ and $d \cdot a \geq \frac{e}{b}$.

(196) Suppose $b < 0$ and $d > 0$ or $b > 0$ and $d < 0$. Then
(i) if $a \cdot b \leq \frac{e}{d}$ or $b \cdot a \leq \frac{e}{d}$, then $a \cdot d \geq \frac{e}{b}$ and $d \cdot a \geq \frac{e}{b}$,
(ii) if $a \cdot b \geq \frac{e}{d}$ or $b \cdot a \geq \frac{e}{d}$, then $a \cdot d \leq \frac{e}{b}$ and $d \cdot a \leq \frac{e}{b}$.

(197) Suppose $0 < a$ or $0 \leq a$ but $a < b$ or $a \leq b$ but $0 < e$ or $0 \leq e$ and $e \leq d$. Then $a \cdot e \leq b \cdot d$ and $a \cdot e \leq d \cdot b$ and $e \cdot a \leq d \cdot b$ and $e \cdot a \leq b \cdot d$.

(198) Suppose $0 > a$ or $0 \geq a$ but $a > b$ or $a \geq b$ but $0 > e$ or $0 \geq e$ and $e \geq d$. Then $a \cdot e \leq b \cdot d$ and $a \cdot e \leq d \cdot b$ and $e \cdot a \leq d \cdot b$ and $e \cdot a \leq b \cdot d$.

(199) Suppose $0 < a$ but $a \leq b$ or $a < b$ and $0 < e$ and $e < d$ or $0 > a$ but $a \geq b$ or $a > b$ and $0 > e$ and $e > d$. Then $a \cdot e < b \cdot d$ and $a \cdot e < d \cdot b$ and $e \cdot a < d \cdot b$ and $e \cdot a < b \cdot d$.

(200) If $e > 0$ but $a > 0$ or $b < 0$ and $a < b$, then $\frac{e}{a} > \frac{e}{b}$.

(201) If $e > 0$ or $e \geq 0$ but $a > 0$ or $b < 0$ and $a \leq b$, then $\frac{e}{a} \geq \frac{e}{b}$.

(202) If $e < 0$ but $a > 0$ or $b < 0$ and $a < b$, then $\frac{e}{a} < \frac{e}{b}$.

(203) If $e < 0$ or $e \leq 0$ but $a > 0$ or $b < 0$ and $a \leq b$, then $\frac{e}{a} \leq \frac{e}{b}$.

Next we state the proposition

(204) For all subsets X, Y of $\mathbb{R}$ such that $X \neq \emptyset$ and $Y \neq \emptyset$ and for all a, b such that $a \in X$ and $b \in Y$ holds $a \leq b$ there exists d such that for every a such that $a \in X$ holds $a \leq d$ and for every b such that $b \in Y$ holds $d \leq b$.

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Received September 5, 1990