FORMALIZED MATHEMATICS
Vol.2, No.**2**, March–April 1991
Université Catholique de Louvain

Calculus of Quantifiers. Deduction Theorem

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Summary. Some tautologies of the Classical Quantifier Calculus. The deduction theorem is also proved.

MML Identifier: `CQC_THE2`.

The papers [11], [13], [8], [2], [5], [3], [12], [10], [9], [1], [6], [4], and [7] provide the terminology and notation for this paper. For simplicity we adopt the following convention: X will denote a subset of CQC$-$WFF, F, G, p, q, r will denote elements of CQC$-$WFF, s, h will denote formulae, and x, y will denote bound variables. Next we state a number of propositions:

(1) If $\vdash p \Rightarrow (q \Rightarrow r)$, then $\vdash p \wedge q \Rightarrow r$.

(2) If $\vdash p \Rightarrow (q \Rightarrow r)$, then $\vdash q \wedge p \Rightarrow r$.

(3) If $\vdash p \wedge q \Rightarrow r$, then $\vdash p \Rightarrow (q \Rightarrow r)$.

(4) If $\vdash p \wedge q \Rightarrow r$, then $\vdash q \Rightarrow (p \Rightarrow r)$.

(5) $y \in \mathrm{snb}(\forall_x s)$ if and only if $y \in \mathrm{snb}(s)$ and $y \neq x$.

(6) $y \in \mathrm{snb}(\exists_x s)$ if and only if $y \in \mathrm{snb}(s)$ and $y \neq x$.

(7) $y \in \mathrm{snb}(s \Rightarrow h)$ if and only if $y \in \mathrm{snb}(s)$ or $y \in \mathrm{snb}(h)$.

(8) $y \in \mathrm{snb}(\neg s)$ if and only if $y \in \mathrm{snb}(s)$.

(9) $y \in \mathrm{snb}(s \wedge h)$ if and only if $y \in \mathrm{snb}(s)$ or $y \in \mathrm{snb}(h)$.

(10) $y \in \mathrm{snb}(s \vee h)$ if and only if $y \in \mathrm{snb}(s)$ or $y \in \mathrm{snb}(h)$.

(11) $x \notin \mathrm{snb}(\forall_{x,y} s)$ and $y \notin \mathrm{snb}(\forall_{x,y} s)$.

(12) $x \notin \mathrm{snb}(\exists_{x,y} s)$ and $y \notin \mathrm{snb}(\exists_{x,y} s)$.

(13) If F is closed, then $x \notin \mathrm{snb}(F)$.

(14) $s \Rightarrow h(x) = (s(x)) \Rightarrow (h(x))$.

(15) $s \vee h(x) = (s(x)) \vee (h(x))$.

ISSN 0777–4028

(16) $\exists_x p(x) = \exists_x p$.

(17) If $x \neq y$, then $\exists_x p(y) = \exists_x (p(y))$.

(18) $\vdash p \Rightarrow \exists_x p$.

(19) If $\vdash p$, then $\vdash \exists_x p$.

(20) $\vdash \forall_x p \Rightarrow \exists_x p$.

(21) $\vdash \forall_x p \Rightarrow \exists_y p$.

(22) If $\vdash p \Rightarrow q$ and $x \notin \mathrm{snb}(q)$, then $\vdash (\exists_x p) \Rightarrow q$.

(23) If $x \notin \mathrm{snb}(p)$, then $\vdash (\exists_x p) \Rightarrow p$.

(24) If $x \notin \mathrm{snb}(p)$ and $\vdash \exists_x p$, then $\vdash p$.

(25) If $p = h(x)$ and $q = h(y)$ and $y \notin \mathrm{snb}(h)$, then $\vdash p \Rightarrow \exists_y q$.

(26) If $\vdash p$, then $\vdash \forall_x p$.

(27) If $x \notin \mathrm{snb}(p)$, then $\vdash p \Rightarrow \forall_x p$.

(28) If $p = h(x)$ and $q = h(y)$ and $x \notin \mathrm{snb}(h)$, then $\vdash \forall_x p \Rightarrow q$.

(29) If $y \notin \mathrm{snb}(p)$, then $\vdash \forall_x p \Rightarrow \forall_y p$.

(30) If $p = h(x)$ and $q = h(y)$ and $x \notin \mathrm{snb}(h)$ and $y \notin \mathrm{snb}(p)$, then $\vdash \forall_x p \Rightarrow \forall_y q$.

(31) If $x \notin \mathrm{snb}(p)$, then $\vdash (\exists_x p) \Rightarrow \exists_y p$.

One can prove the following propositions:

(32) If $p = h(x)$ and $q = h(y)$ and $x \notin \mathrm{snb}(q)$ and $y \notin \mathrm{snb}(h)$, then $\vdash (\exists_x p) \Rightarrow \exists_y q$.

(34)[1] $\vdash \forall_x (p \Rightarrow q) \Rightarrow (\forall_x p \Rightarrow \forall_x q)$.

(35) If $\vdash \forall_x (p \Rightarrow q)$, then $\vdash \forall_x p \Rightarrow \forall_x q$.

(36) $\vdash \forall_x (p \Leftrightarrow q) \Rightarrow (\forall_x p \Leftrightarrow \forall_x q)$.

(37) If $\vdash \forall_x (p \Leftrightarrow q)$, then $\vdash \forall_x p \Leftrightarrow \forall_x q$.

(38) $\vdash \forall_x (p \Rightarrow q) \Rightarrow ((\exists_x p) \Rightarrow \exists_x q)$.

(39) If $\vdash \forall_x (p \Rightarrow q)$, then $\vdash (\exists_x p) \Rightarrow \exists_x q$.

(40) $\vdash \forall_x (p \wedge q) \Rightarrow \forall_x p \wedge \forall_x q$ and $\vdash \forall_x p \wedge \forall_x q \Rightarrow \forall_x (p \wedge q)$.

(41) $\vdash \forall_x (p \wedge q) \Leftrightarrow \forall_x p \wedge \forall_x q$.

(42) $\vdash \forall_x (p \wedge q)$ if and only if $\vdash \forall_x p \wedge \forall_x q$.

(43) $\vdash \forall_x p \vee \forall_x q \Rightarrow \forall_x (p \vee q)$.

(44) $\vdash (\exists_x p \vee q) \Rightarrow (\exists_x p) \vee \exists_x q$ and $\vdash (\exists_x p) \vee \exists_x q \Rightarrow \exists_x p \vee q$.

(45) $\vdash (\exists_x p \vee q) \Leftrightarrow (\exists_x p) \vee \exists_x q$.

(46) $\vdash \exists_x p \vee q$ if and only if $\vdash (\exists_x p) \vee \exists_x q$.

(47) $\vdash (\exists_x p \wedge q) \Rightarrow (\exists_x p) \wedge \exists_x q$.

(48) If $\vdash \exists_x p \wedge q$, then $\vdash (\exists_x p) \wedge \exists_x q$.

(49) $\vdash \forall_x \neg\neg p \Rightarrow \forall_x p$ and $\vdash \forall_x p \Rightarrow \forall_x \neg\neg p$.

(50) $\vdash \forall_x \neg\neg p \Leftrightarrow \forall_x p$.

(51) $\vdash (\exists_x \neg\neg p) \Rightarrow \exists_x p$ and $\vdash (\exists_x p) \Rightarrow \exists_x \neg\neg p$.

[1]The proposition (33) was either repeated or obvious.

(52) $\vdash (\exists_x \neg\neg p) \Leftrightarrow \exists_x p$.
(53) $\vdash \neg\exists_x \neg p \Rightarrow \forall_x p$ and $\vdash \forall_x p \Rightarrow \neg\exists_x \neg p$.
(54) $\vdash \neg\exists_x \neg p \Leftrightarrow \forall_x p$.
(55) $\vdash \neg\forall_x p \Rightarrow \exists_x \neg p$ and $\vdash (\exists_x \neg p) \Rightarrow \neg\forall_x p$.
(56) $\vdash \neg\forall_x p \Leftrightarrow \exists_x \neg p$.
(57) $\vdash \neg\exists_x p \Rightarrow \forall_x \neg p$ and $\vdash \forall_x \neg p \Rightarrow \neg\exists_x p$.
(58) $\vdash \forall_x \neg p \Leftrightarrow \neg\exists_x p$.
(59) $\vdash \forall_x \forall_y p \Rightarrow \forall_y \forall_x p$ and $\vdash \forall_{x,y} p \Rightarrow \forall_{y,x} p$.
(60) If $p = h(x)$ and $q = h(y)$ and $y \notin \mathrm{snb}(h)$, then $\vdash \forall_x \forall_y q \Rightarrow \forall_x p$.
(61) $\vdash (\exists_x \exists_y p) \Rightarrow \exists_y \exists_x p$ and $\vdash (\exists_{x,y} p) \Rightarrow (\exists_{y,x} p)$.
(62) If $p = h(x)$ and $q = h(y)$ and $y \notin \mathrm{snb}(h)$, then $\vdash (\exists_x p) \Rightarrow (\exists_{x,y} q)$.

We now state a number of propositions:

(63) $\vdash (\exists_x \forall_y p) \Rightarrow \forall_y \exists_x p$.
(64) $\vdash \exists_x p \Leftrightarrow p$.
(65) $\vdash (\exists_x p \Rightarrow q) \Rightarrow (\forall_x p \Rightarrow \exists_x q)$ and $\vdash (\forall_x p \Rightarrow \exists_x q) \Rightarrow \exists_x p \Rightarrow q$.
(66) $\vdash (\exists_x p \Rightarrow q) \Leftrightarrow (\forall_x p \Rightarrow \exists_x q)$.
(67) $\vdash \exists_x p \Rightarrow q$ if and only if $\vdash \forall_x p \Rightarrow \exists_x q$.
(68) $\vdash \forall_x (p \wedge q) \Rightarrow p \wedge \forall_x q$.
(69) $\vdash \forall_x (p \wedge q) \Rightarrow \forall_x p \wedge q$.
(70) If $x \notin \mathrm{snb}(p)$, then $\vdash p \wedge \forall_x q \Rightarrow \forall_x (p \wedge q)$.
(71) If $x \notin \mathrm{snb}(p)$ and $\vdash p \wedge \forall_x q$, then $\vdash \forall_x (p \wedge q)$.
(72) If $x \notin \mathrm{snb}(p)$, then $\vdash p \vee \forall_x q \Rightarrow \forall_x (p \vee q)$ and $\vdash \forall_x (p \vee q) \Rightarrow p \vee \forall_x q$.
(73) If $x \notin \mathrm{snb}(p)$, then $\vdash p \vee \forall_x q \Leftrightarrow \forall_x (p \vee q)$.
(74) If $x \notin \mathrm{snb}(p)$, then $\vdash p \vee \forall_x q$ if and only if $\vdash \forall_x (p \vee q)$.
(75) If $x \notin \mathrm{snb}(p)$, then $\vdash p \wedge \exists_x q \Rightarrow \exists_x p \wedge q$ and $\vdash (\exists_x p \wedge q) \Rightarrow p \wedge \exists_x q$.
(76) If $x \notin \mathrm{snb}(p)$, then $\vdash p \wedge \exists_x q \Leftrightarrow \exists_x p \wedge q$.
(77) If $x \notin \mathrm{snb}(p)$, then $\vdash p \wedge \exists_x q$ if and only if $\vdash \exists_x p \wedge q$.
(78) If $x \notin \mathrm{snb}(p)$, then $\vdash \forall_x (p \Rightarrow q) \Rightarrow (p \Rightarrow \forall_x q)$ and $\vdash (p \Rightarrow \forall_x q) \Rightarrow \forall_x (p \Rightarrow q)$.
(79) If $x \notin \mathrm{snb}(p)$, then $\vdash (p \Rightarrow \forall_x q) \Leftrightarrow \forall_x (p \Rightarrow q)$.
(80) If $x \notin \mathrm{snb}(p)$, then $\vdash \forall_x (p \Rightarrow q)$ if and only if $\vdash p \Rightarrow \forall_x q$.
(81) If $x \notin \mathrm{snb}(q)$, then $\vdash (\exists_x p \Rightarrow q) \Rightarrow (\forall_x p \Rightarrow q)$.
(82) $\vdash (\forall_x p \Rightarrow q) \Rightarrow \exists_x p \Rightarrow q$.
(83) If $x \notin \mathrm{snb}(q)$, then $\vdash \forall_x p \Rightarrow q$ if and only if $\vdash \exists_x p \Rightarrow q$.
(84) If $x \notin \mathrm{snb}(q)$, then $\vdash ((\exists_x p) \Rightarrow q) \Rightarrow \forall_x (p \Rightarrow q)$ and $\vdash \forall_x (p \Rightarrow q) \Rightarrow ((\exists_x p) \Rightarrow q)$.
(85) If $x \notin \mathrm{snb}(q)$, then $\vdash ((\exists_x p) \Rightarrow q) \Leftrightarrow \forall_x (p \Rightarrow q)$.
(86) If $x \notin \mathrm{snb}(q)$, then $\vdash (\exists_x p) \Rightarrow q$ if and only if $\vdash \forall_x (p \Rightarrow q)$.
(87) If $x \notin \mathrm{snb}(p)$, then $\vdash (\exists_x p \Rightarrow q) \Rightarrow (p \Rightarrow \exists_x q)$.

(88) $\vdash (p \Rightarrow \exists_x q) \Rightarrow \exists_x p \Rightarrow q$.

(89) If $x \notin \mathrm{snb}(p)$, then $\vdash (p \Rightarrow \exists_x q) \Leftrightarrow \exists_x p \Rightarrow q$.

(90) If $x \notin \mathrm{snb}(p)$, then $\vdash p \Rightarrow \exists_x q$ if and only if $\vdash \exists_x p \Rightarrow q$.

(91) $\{p\} \vdash p$.

(92) $\mathrm{Cn}(\{p\} \cup \{q\}) = \mathrm{Cn}\{p \wedge q\}$.

(93) $\{p, q\} \vdash r$ if and only if $\{p \wedge q\} \vdash r$.

The following propositions are true:

(94) If $X \vdash p$, then $X \vdash \forall_x p$.

(95) If $x \notin \mathrm{snb}(p)$, then $X \vdash \forall_x (p \Rightarrow q) \Rightarrow (p \Rightarrow \forall_x q)$.

(96) If F is closed and $X \cup \{F\} \vdash G$, then $X \vdash F \Rightarrow G$.

References

[1] Grzegorz Bancerek. Connectives and subformulae of the first order language. *Formalized Mathematics*, 1(**3**):451–458, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Czesław Byliński and Grzegorz Bancerek. Variables in formulae of the first order language. *Formalized Mathematics*, 1(**3**):459–469, 1990.

[7] Agata Darmochwał. A first–order predicate calculus. *Formalized Mathematics*, 1(**4**):689–695, 1990.

[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[9] Piotr Rudnicki and Andrzej Trybulec. A first order language. *Formalized Mathematics*, 1(**2**):303–311, 1990.

[10] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[12] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[13] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received October 24, 1990