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Submodules and Cosets of Submodules in Left Module over Associative Ring [1]

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Summary. Notions of Submodules in Left Module over Associative Ring and Cosets of Submodules in Left Module over Associative Ring. A few basic theorems related to these notions are proved. This article originated as a generalization of the article [12].

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The notation and terminology used here are introduced in the following articles: [8], [2], [14], [13], [10], [11], [7], [1], [3], [9], [4], [6], and [5]. For simplicity we follow a convention: x will be arbitrary, R will be an associative ring, a will be a scalar of R, V, X, Y will be left modules over R, and u, v, v_1, v_2 will be vectors of V. Let us consider R, V. A subset of V is a subset of the carrier of the carrier of V.

In the sequel V_1, V_2, V_3 will denote subsets of V. Let us consider R, V, V_1. We say that V_1 is linearly closed if and only if:

(Def.1) for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

We now state a number of propositions:

(1) If for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$, then V_1 is linearly closed.

(2) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$.

(3) If V_1 is linearly closed, then for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

(4) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then $\Theta_V \in V_1$.

(5) If V_1 is linearly closed, then for every v such that $v \in V_1$ holds $-v \in V_1$.

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(6) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v - u \in V_1$.

(7) $\{\Theta_V\}$ is linearly closed.

(8) If the carrier of the carrier of $V = V_1$, then V_1 is linearly closed.

(9) If V_1 is linearly closed and V_2 is linearly closed and $V_3 = \{v + u : v \in V_1 \wedge u \in V_2\}$, then V_3 is linearly closed.

(10) If V_1 is linearly closed and V_2 is linearly closed, then $V_1 \cap V_2$ is linearly closed.

Let us consider R, V. A left module over R is called a submodule of V if:

(Def.2) the carrier of the carrier of it $\subseteq$ the carrier of the carrier of V and the zero of the carrier of it $=$ the zero of the carrier of V and the addition of the carrier of it $=$ (the addition of the carrier of V) $\restriction [\![$ the carrier of the carrier of it, the carrier of the carrier of it $]\!]$ and the left multiplication of it $=$ (the left multiplication of V) $\restriction [\![$ the carrier of R, the carrier of the carrier of it $]\!]$.

We now state the proposition

(11) If the carrier of the carrier of $X \subseteq$ the carrier of the carrier of V and the zero of the carrier of $X =$ the zero of the carrier of V and the addition of the carrier of $X =$ (the addition of the carrier of V) $\restriction [\![$ the carrier of the carrier of X, the carrier of the carrier of X $]\!]$ and the left multiplication of $X =$ (the left multiplication of V) $\restriction [\![$ the carrier of R, the carrier of the carrier of X $]\!]$, then X is a submodule of V.

We follow a convention: W, W_1, W_2 denote submodules of V and w, w_1, w_2 denote vectors of W. The following propositions are true:

(12) The carrier of the carrier of $W \subseteq$ the carrier of the carrier of V.

(13) The zero of the carrier of $W =$ the zero of the carrier of V.

(14) The addition of the carrier of $W =$ (the addition of the carrier of V) $\restriction [\![$ the carrier of the carrier of W, the carrier of the carrier of W $]\!]$.

(15) The left multiplication of $W =$ (the left multiplication of V) $\restriction [\![$ the carrier of R, the carrier of the carrier of W $]\!]$.

(16) If $x \in W_1$ and W_1 is a submodule of W_2, then $x \in W_2$.

(17) If $x \in W$, then $x \in V$.

(18) w is a vector of V.

(19) $\Theta_W = \Theta_V$.

(20) $\Theta_{W_1} = \Theta_{W_2}$.

(21) If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(22) If $w = v$, then $a \cdot w = a \cdot v$.

(23) If $w = v$, then $-v = -w$.

(24) If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(25) $\Theta_V \in W$.

(26) $\Theta_{W_1} \in W_2$.

(27) $\Theta_W \in V$.

(28) If $u \in W$ and $v \in W$, then $u + v \in W$.

(29) If $v \in W$, then $a \cdot v \in W$.

(30) If $v \in W$, then $-v \in W$.

(31) If $u \in W$ and $v \in W$, then $u - v \in W$.

(32) V is a submodule of V.

(33) If V is a submodule of X and X is a submodule of V, then $V = X$.

(34) If V is a submodule of X and X is a submodule of Y, then V is a submodule of Y.

(35) If the carrier of the carrier of $W_1 \subseteq$ the carrier of the carrier of W_2, then W_1 is a submodule of W_2.

(36) If for every v such that $v \in W_1$ holds $v \in W_2$, then W_1 is a submodule of W_2.

(37) If the carrier of the carrier of $W_1 =$ the carrier of the carrier of W_2, then $W_1 = W_2$.

(38) If for every v holds $v \in W_1$ if and only if $v \in W_2$, then $W_1 = W_2$.

(39) If the carrier of the carrier of $W =$ the carrier of the carrier of V, then $W = V$.

(40) If for every v holds $v \in W$, then $W = V$.

(41) If the carrier of the carrier of $W = V_1$, then V_1 is linearly closed.

(42) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then there exists W such that $V_1 =$ the carrier of the carrier of W.

Let us consider R, V. The functor $\mathbf{0}_V$ yields a submodule of V and is defined as follows:

(Def.3) the carrier of the carrier of $\mathbf{0}_V = \{\Theta_V\}$.

Let us consider R, V. The functor Ω_V yielding a submodule of V is defined by:

(Def.4) $\Omega_V = V$.

The following propositions are true:

(43) The carrier of the carrier of $\mathbf{0}_V = \{\Theta_V\}$.

(44) If the carrier of the carrier of $W = \{\Theta_V\}$, then $W = \mathbf{0}_V$.

(45) $\Omega_V = V$.

(46) $x \in \mathbf{0}_V$ if and only if $x = \Theta_V$.

(47) $\mathbf{0}_W = \mathbf{0}_V$.

(48) $\mathbf{0}_{W_1} = \mathbf{0}_{W_2}$.

(49) $\mathbf{0}_W$ is a submodule of V.

(50) $\mathbf{0}_V$ is a submodule of W.

(51) $\mathbf{0}_{W_1}$ is a submodule of W_2.

(52) W is a submodule of Ω_V.

(53) V is a submodule of Ω_V.

Let us consider R, V, v, W. The functor $v + W$ yields a subset of V and is defined by:

(Def.5) $v + W = \{v + u : u \in W\}$.

Let us consider R, V, W. A subset of V is said to be a coset of W if:

(Def.6) there exists v such that it $= v + W$.

In the sequel B, C are cosets of W. One can prove the following propositions:

(54) $v + W = \{v + u : u \in W\}$.

(55) There exists v such that $C = v + W$.

(56) If $V_1 = v + W$, then V_1 is a coset of W.

(57) $x \in v + W$ if and only if there exists u such that $u \in W$ and $x = v + u$.

(58) $\Theta_V \in v + W$ if and only if $v \in W$.

(59) $v \in v + W$.

(60) $\Theta_V + W =$ the carrier of the carrier of W.

(61) $v + \mathbf{0}_V = \{v\}$.

(62) $v + \Omega_V =$ the carrier of the carrier of V.

(63) $\Theta_V \in v + W$ if and only if $v + W =$ the carrier of the carrier of W.

(64) $v \in W$ if and only if $v + W =$ the carrier of the carrier of W.

(65) If $v \in W$, then $a \cdot v + W =$ the carrier of the carrier of W.

(66) $u \in W$ if and only if $v + W = v + u + W$.

(67) $u \in W$ if and only if $v + W = (v - u) + W$.

(68) $v \in u + W$ if and only if $u + W = v + W$.

(69) If $u \in v_1 + W$ and $u \in v_2 + W$, then $v_1 + W = v_2 + W$.

(70) If $v \in W$, then $a \cdot v \in v + W$.

(71) If $v \in W$, then $-v \in v + W$.

(72) $u + v \in v + W$ if and only if $u \in W$.

(73) $v - u \in v + W$ if and only if $u \in W$.

(74) $u \in v+W$ if and only if there exists v_1 such that $v_1 \in W$ and $u = v+v_1$.

(75) $u \in v+W$ if and only if there exists v_1 such that $v_1 \in W$ and $u = v-v_1$.

(76) There exists v such that $v_1 \in v + W$ and $v_2 \in v + W$ if and only if $v_1 - v_2 \in W$.

(77) If $v+W = u+W$, then there exists v_1 such that $v_1 \in W$ and $v+v_1 = u$.

(78) If $v+W = u+W$, then there exists v_1 such that $v_1 \in W$ and $v-v_1 = u$.

(79) $v + W_1 = v + W_2$ if and only if $W_1 = W_2$.

(80) If $v + W_1 = u + W_2$, then $W_1 = W_2$.

In the sequel C_1 denotes a coset of W_1 and C_2 denotes a coset of W_2. Next we state a number of propositions:

(81) There exists C such that $v \in C$.

(82) C is linearly closed if and only if $C =$ the carrier of the carrier of W.

(83) If $C_1 = C_2$, then $W_1 = W_2$.

(84) $\{v\}$ is a coset of $\mathbf{0}_V$.

(85) If V_1 is a coset of $\mathbf{0}_V$, then there exists v such that $V_1 = \{v\}$.

(86) The carrier of the carrier of W is a coset of W.

(87) The carrier of the carrier of V is a coset of Ω_V.

(88) If V_1 is a coset of Ω_V, then $V_1 =$ the carrier of the carrier of V.

(89) $\Theta_V \in C$ if and only if $C =$ the carrier of the carrier of W.

(90) $u \in C$ if and only if $C = u + W$.

(91) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u+v_1 = v$.

(92) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u-v_1 = v$.

(93) There exists C such that $v_1 \in C$ and $v_2 \in C$ if and only if $v_1 - v_2 \in W$.

(94) If $u \in B$ and $u \in C$, then $B = C$.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[4] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.

[5] Michał Muzalewski and Wojciech Skaba. Finite sums of vectors in left module over associative ring. *Formalized Mathematics*, 2(**2**):279–282, 1991.

[6] Michał Muzalewski and Lesław W. Szczerba. Construction of finite sequences over ring and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):97–104, 1991.

[7] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[9] Wojciech A. Trybulec. Finite sums of vectors in vector space. *Formalized Mathematics*, 1(**5**):851–854, 1990.

[10] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.

[11] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Formalized Mathematics*, 1(**5**):855–864, 1990.

[12] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Formalized Mathematics*, 1(**5**):865–870, 1990.

[13] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[14] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

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